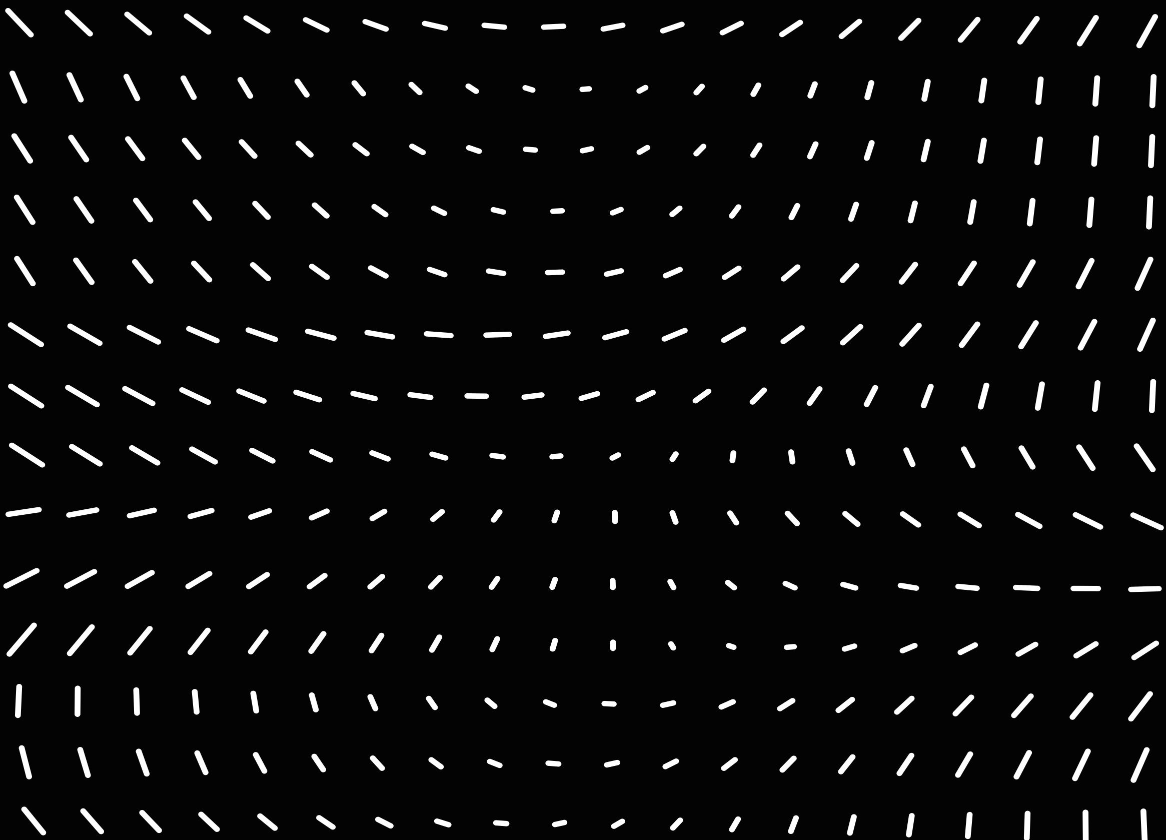
STAT 268

Lecture Notes



Topic 1

Topic 1: Probability

Probability is a measure of one's belief in the occurance of a random event.

Random Event: outcome of experiments that cannot be determined uniquely from known conditions

Eg. Coin Flip $H = 0.5$
$T = 0.5$

Classical prob. often defined through relative frequency of occurance of a random event

Buffon's. 4040 tosses, 2048 heads

Relative Frequency $\frac{2048}{4040} = 0.507$

Karl Pearson 24 000 tosses, 12012 heads

Need to count outcomes for an experiment
- Counting Rules
- multiplicative Rules

↳ We can do job 1 in 20 ways
We can do job 2 in 7 ways
We can do job 1 and job 2 in $n \times m$ ways

E.g Tossing 2 Dice, 1 green, 1 red

\# possible outcomes: $6 \times 6 = 36$

All outcomes: $\{ (1,1), (1,2), \dots (1,6)$
$(2,1), (2,2) \dots (2,6)$
$\dots (6,6) \}$

↓
order matters

Problem 1-1

a) $5^3 = 125$

Addition Rule:

Suppose we can do job 1 in n ways and job 2 in x ways. Then we can do either job 1 or job 2 in $x+n$ ways. → ?

E.g: Tossing a pair of dice

\# of possible outcomes s.t { the total # of pts on the 2 dice is >10 }

$r+g=11 \rightarrow (5,6), (6,5)$ - 2 ways
$r+g=12 \rightarrow (6,6)$ - 1 way

$2+1=3$ ways

3. Permutation Rule:

Suppose there are n distinct objects
The # of ways to arrange (permute) r objects selected from these n objects.

is $P_1^n = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$

$P_n^r = n!$

$P_1^n = n$ objects are drawn without replacement.

Note: **Order Matters**

Problem 1-1 a) A B C
5 5 5

b) A B C $\quad 5P3 = 60$ ways
1 2 3
2 1 3

Extension.
\# of distinguishable permutation of the word "MISS"

MISS $\quad \frac{4!}{2!}$

MS_1IS_2 } same outcome
MS_2IS_1

\# of Distinguishable ways for "MISSISSAUGA"

$$\frac{11!}{2!\cdot4!\cdot2!} = 415800$$ ways

Combinations.

From n distinct objects the # of ways to take out combinations of r subjects at a time is denoted by C^n_r

n- total
r- subset

$$\binom{n}{r} = \frac{P^n_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

order does Not matter

Objects are drawn without replacement.

$\binom{n}{n} = 1 \quad \binom{n}{1} = n$

$\binom{n}{r} \cdot \binom{n}{n-r}$

$\binom{8}{5} = \binom{8}{3}$

Problem 1-2

Total 4 chem, 3 Physc

Choose 2 chem, 1 physc

$4C2 \times 3C1 = 18$ ways

- Probability Concepts S2.2-2-4

1. Sample space (S)

   set of all possible outcomes of an experiment.

   Each outcome in S, is called a Sample point (Potential outcome)

2. Random Event (denoted by A, B...)

   a subset of a sample space.

An event A is said to occur if the outcome of the experiment belongs to A.

Problem 1-3

a) S = {HHH, HMH, HHM, MHH, MHM, MMH, MHM, MMM}

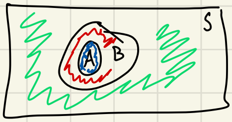
8 outcomes

b) A = {HMH, HHM, MHH}

B = {MMM, HMM, MHM, MHM}

Review of set notation

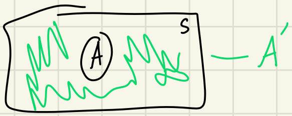
A ⊂ B - A is a subset of B



A ∪ B Union of A and B

A or B or Both

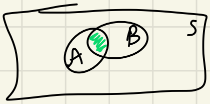
A' or $\bar{A}$: The complement of A.



$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cup B'$

A ∩ B Intersection Of A and B



Both A and B

3.) Probability

Suppose S is a sample space of an experiment. The probability of an event A. $P(A)$, must satisfy:

Axiom 1: $P(A) \geq 0$
Axiom 2: $P(S) = 1$
Axiom 3: If $A_1, A_2, A_3 \ldots$ are pairwise mutually exclusive events in S.

Definition of Probability

$$A_i \cap A_J = P \text{ if } i \neq j, \text{ then}$$
$$P(A_1 \cup A_2 \cup A_3 \ldots) = \sum_{T=1}^{\infty} P(A_i)$$

Note: Assigned Probability must satisfy Axioms 1, 2, 3 and be realistic.

A_1, A_2 are disjoint events,
Axiom 3 $\Rightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Calculating Point of an event using Sample-Point Method:

Thm. 2.2 An experiment can give n different outcomes, each outcome is equally likely. Event A consists of m of these outcomes.

Then $P(A) = \dfrac{m}{n} = \dfrac{\#\text{ of sample pts in } A}{\# \ldots \text{ in } S}$

Problem 1.4

Canidates : $\{1, 2, 3, 4, 5\}$
Best Poorest

$S = \{$

$\binom{5}{2} = 10$

$A = \{(1,4), (1,5)\}$

$= \frac{2}{10}$ or 0.2 or 20%

$B = \{(1,2),(1,3) \dots \}$

$\frac{7}{10} = 0.7$

Problem 1.5

$A = \{$a full house$)$

$\binom{52}{5} = 2598960$

13 face value options

$\binom{4}{3}$ 3 out of 4

$$P(A) = \frac{13 \cdot \binom{4}{3} \cdot 12 \binom{4}{2}}{\binom{52}{5}}$$

$$P(A) = \frac{3744}{2598960}$$

$P(A) = 0.000144$

b) Total # of Arrangements of 52 cards cards positions:

$$P(A) = \frac{13 \cdot}{52!}$$

Thm 2.1: If an experiment gives a discrete sample space S, the prob of an event A ($A \subseteq S$) is the sum of the probabilities of all sample pts comprising A.

Note: the sample pts in S may or may not be equally likely.

Ex. A coin is flipped untill heads appears

$S = \{\underset{p_1}{H}, \underset{p_2}{TH}, \underset{p_3}{TTH}, \dots\}$

② let $p(o_i) = (\frac{1}{2})^i$, $i = 1, 2, 3$

This defines a prob measure for this experiment. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots =$

Review Geometric Series:

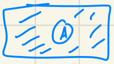
$$a + ar + ar^2 \dots ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a}{1-r}$$

if $0 > x > 1$

Rules of Probability (2.5)

1) Thm 2.3 $P(A') = 1 - P(A)$



Pf $S = A \cup A'$

$A \cap A' = \emptyset$

∴ A and A' are mutually exclusive.

Axiom 3: $P(A \cup A') = P(A) + P(A')$

$P(S) = 1$

Axiom 2: $P(A) + P(A') = 1$

The Birthday Problem

Prt-6 The # of sample pts in S

day 1 □ day 2 □ person 20 □

$365 \cdot 365 \quad 365 = 365^{20}$

A = {at least 2 people have the same bday}

A' = {no person's same bday}

sample pt # in A' $\quad 365 \cdot 364 \dots \underset{346}{(365-20+1)} = P^{365}_{20}$

$$P(A') = \frac{P^{365}_{20}}{365^{20}} = 0.5886$$

$P(A) = 1 - P(A') = 1 - 0.5886$
$= 0.4114$

b) P(no one is born on Sept. 1) $= \frac{364^{20}}{365^{20}}$

P(1 was born on Sept. 1) $= \frac{20 \cdot 364^{19}}{365^{20}}$

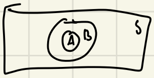
Let A represent at least 2 born on Sept. 1.

$$P(A) = 1 - \frac{364^{20}}{366^{20}} = 0.0014$$

Other rules of Probability rules

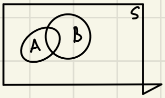
2) $P(\emptyset) = 0$ for any Sample Space S.

3) A, B are events is sample space S,
and $A \subset B$, then $P(A) \le P(B)$



4.) $0 \le P(A) \le 1$, for any event A.

5.) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



6.) $P(A \cup B \cup C) - P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

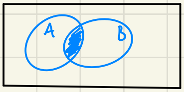
Pr. 1.8

$A = \{\text{male}\}$

$B = \{\text{married}\}$

Find $P(A \cap B)$ Know $P(A) = 0.58$, $P(B) = 0.65$

$P(A' \cap B') = 0.25$



By Morgans Rule

$A' \cap B' = (A \cup B)'$

$P(A' \cap B) = 0.25$

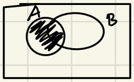
$P(A \cup B) = 1 - 0.25 = 0.75$

$P(A \cup B) = P(A) + P(B) - P(A \cup B)$

$= 0.58 + 0.65 - 0.75$

$P(A \cup B) = 0.48$

b) $P(A \cap B') = P(A) - P(A \cap B)$



$= 0.58 - 0.48$

$= 0.1$

$((A \cap B) \cup (A \cap B') = A$

disjoint

Pr 1.9

Method 1: order matters

$P(A) = P(A_1) + P(A_2)$

$= P(\text{both } W \text{ on ends}) + P(\text{both } M \text{ on ends})$

$$= \frac{P^3_2 \cdot 6!}{8!} + \frac{P^5_2 \cdot 6!}{8!}$$

$$= \frac{P^3_2}{P^8_2} = \frac{P^3_2}{\frac{8!}{6!}} + \frac{P^5_2}{P^8_2} = 13/28$$

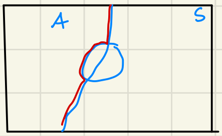
Method 2: order does not matter

$$P(A) = \frac{\binom{3}{2}}{\binom{8}{2}} + \frac{\binom{5}{2}}{\binom{8}{2}} = 13/28$$

Conditional Probability

If $P(A) \neq 0$ then the Conditional Prob of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



Pr 1.10

C = {color blind}

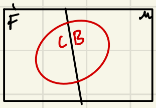
M = {Male}

$P(C) = P(C \cap M) + P(C \cap M')$

$= 0.05 + 0.005 = 0.055$

(b) $P(C|M)$

$$= \frac{P(C \cap M)}{P(M)}$$



$$= \frac{0.05}{0.5} = 0.1$$

c) 0.01

Example: Toss a die onced

Given, the number is higher than 3, Whats Probability is even.

possible: {4, 5, 6}

$\therefore P = 0.66$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{3/6} = \frac{2}{3}$$

Solve (a) when the die is twice as likely to occur as each even #.

Assign Prob to an even #

$2w + w + 2w + w + 2w + w = 1$

$w = \frac{1}{9}$

$S = \{1, 2, 3$

$$\text{Probs} = \frac{2}{9}, \frac{1}{9}, \frac{2}{9} = \frac{P(A \cap B)}{P(A)} = \frac{P\{4, 6\}}{P\{4, 5, 6\}} = \frac{\frac{1}{9} + \frac{1}{9}}{\frac{1}{9} + \frac{2}{9} + \frac{1}{9}}$$

$= \frac{1}{2}$

Multiplication Law of Probability

$P(A \cap B) = P(A) \cdot P(B|A)$

- A and B must be in S.
- A/B cant be 0.

$\downarrow$

Likewise, $P(A \cap B) = P(B) \cdot P(A|B)$

Problem 1.1

A = {both are defective}

225 N.DF

15 DF

Total 245

$240C2 =$

$15C2$ = Defective

$$P(A) = \frac{15C2}{245C2} = 0.0037$$

$$P(B) = \frac{225 \cdot 15}{245C2} = 0.117$$

Method 2:

D_1 = {1st TV. is DF

D_2 = {2nd TV. is DF

a) $P(D_1 \cap D_2) = P(D_1) \cdot P(D_2|D_1)$

$$= \frac{15}{240} \cdot \frac{14}{239} = 0.0037$$

b) P(1 DF, 1 good)

$= P(D_1 \cap D_2') + P(D_1' \cap D_2)$

$= P(D_1) \cdot P(D_2|D_1) + P(D_1') \cdot P(D_2|D_1)$

$$= \frac{15}{240} \cdot \frac{225}{239} + \frac{225}{240} \cdot \frac{15}{239} = 0.1177$$

Independent events (2.7)

Def: Events A, B are independent iff $P(A\cap B) = P(A)\cdot P(B)$

Multiplication Law: $P(A\cap B) = P(A)\cdot P(B|A)$

If A and B are indep. then $P(B) = P(B|A)$

$3! = 6$

Pr 1-12 $S = \{xyz, xzy, yxz, yzx, zxy, zyx\}$

Event A: $3/6$

Event B: $\frac{2}{6}$

Event C: $2/6$

$P(A\cap B) = P(A)\cdot P(B)$

A ∩ B are not indep

There is an outcomes that captures them both.

But A and C are independent

$P(A\cap C) = P(A)\cdot P(C)$

Note: A, B are indep

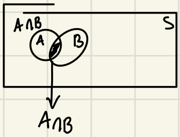
$\Rightarrow$ A and B' are indep

$\Rightarrow$ A' and B are indep

$\Rightarrow$ A' and B' are indep

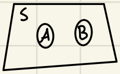
Proof:

Need to show $P(A'\cap B') = P(A)\cdot P(B')$



$$P(A) = P(\underbrace{A\cap B \text{ or } A\cap B'}_{\text{disjoint}})$$
$$= P(A\cup B) + P(A\cap B')$$
$$= P(A)\cdot P(B) + P(A\cap B')$$
$$P(A\cap B') = P(A) - P(A)\cdot P(B)$$
$$= P(A)\left[1 - P(B)\right]$$
$$= P(A)\cdot P(B')$$

Recall, A, B are mutually exclusive iff $A\cap B = \emptyset$ (A, B are not empty)



$P(A\cap B) = 0$

A, B are not independent

Problem 1.3

a) $S = \{AA, AB, BA, BB\}$

$P(E) = P\{AA, AB, BA\} = \frac{2}{3}\cdot\frac{2}{3} + \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = 8/9$

b) $S = \{AAA, ABA, AAB, BAA$

$\underbrace{ABB, BAB, BBA, BBB}\}$

$C = \{B \text{ wins at least twice}\}$

$P(C) = \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{2}{3} + \left(\frac{1}{3}\right)^3 = \frac{7}{27}$

Pairwise independence does not imply mutual independence!

Pr. 1-14 (a) $A = \{HH, HT\}$ $B = \{HH, TH\}$

$C = \{HH, TT\}$

$A\cap B = \{HH\}$

$P(A\cap B) = \frac{1}{4}$. $P(A) = P(B) = \frac{1}{2} \rightarrow P(A\cap B)$

$P(A\cap C) = P(A)\cdot P(C)$

$P(B\cap C) = P(B)\cdot P(C)$

what about $P(A\cap B\cap C) \overset{?}{=} P(A)\cdot P(B)\cdot P(C)$

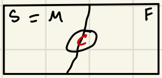
$P(A\cap B\cap C) = \frac{1}{4}$

$P(A), P(B), P(C) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

A, B, C are not mutually independent.

Law of total Probability

Pr. 1.16



$P(C|M)$

$P(C|F)$

$P(C) = P(C\cap M) + P(F\cap C)$

$= P(M)\,P(C|M) + P(F)\cdot P(C|F)$

$= 0.5\cdot 0.1 + 0.5\cdot 0.01$

$= 0.05$

Definition: Let sets $B1, B2 \ldots B_k$ be s.t

1. $B_1\cup B_2\cup \ldots \cup B_k = S$
2. $B_i\cap B_j = \emptyset$ for $i\neq j$ (pairwise mutually exclusive).

Then $B1 - B_k$ is a Partition of S

Thm : 2.12 (Law of Total Prob)

if events $B_1 \dots B_k$ from a partition of S and $P(B_i) \neq 0$ then for any event $A \subset S$

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)$$

pr 1-17

a) $C = \{\text{correct ans}\}$

$K = \{\text{student knows}\}$

$$P(C) = P(C \cap K) + P(C \cap K')$$

$$= P(K) \cdot P(C|K) + P(K') \cdot P(C|K')$$

$$= 0.6 \cdot 1 + 0.4 \cdot \frac{1}{4} = 0.7$$

b) $$P(K|C) = \frac{P(C \cap K)}{P(C)} = \frac{P(K) \cdot P(C|K)}{P(C)}$$

$$= \frac{0.6 \cdot 0.1}{0.7} = \frac{6}{7}$$

$$P(K|C) = \frac{P(K) \cdot P(C|K)}{P(K) \cdot P\underbrace{(C|K) + P(K')}\cdot P(C|K')}$$

Topic 2

Topic 2: Probability Distributions, Densities and Math expressions

Random Variable – function that assigns a numerical value to each outcome

Discrete R.V – Takes on countable number of Possible Values

continous R.V – Takes on an infinite number of Possible values in a range.

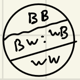
Probability Distribution's for Discrete R.V's

Def: If x is a discrete r.V, then $f(x) = P(X = x)$ for each x in the range of x is called the pmf Probability mass function of x

Pr-2.1

- 5 black
- 3 white

a) S



$x = \{ \# \text{ of black} \}$

$x = 0, 1, 2$

$$f(0) = P\{WW\} = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}$$

$$f(1) = P(x=1) = P\{BW, WB\} = \frac{\binom{5}{1} \cdot \binom{3}{1}}{\binom{8}{2}} = \frac{15}{28}$$

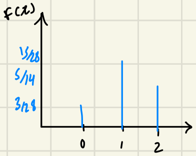
$$f(2) = P(x=2) = P\{BB\} = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{5}{14}$$

To confirm all Possible outcomes = 1

Prof of x:

x	0	1	2
$f(x)$	3/28	15/28	5/14

Graph your PnF



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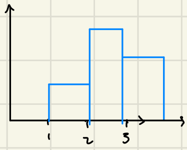


1 b) P(matching pair) = $P(x=0 \text{ or } x=2)$

$$= \frac{3}{28} + \frac{5}{14} = \frac{13}{28}$$

c) $$F(x) = \frac{\binom{5}{x}\binom{3}{2-x}}{\binom{8}{2}} ; \quad x = 0, 1, 2$$

$P(x = x)$

(Hyper geometric distribution)

Thm 3.1. For any discrete R.V x.
Let $F(x)$ be its P mf. Then

1. $0 < f(x) \leq x$, for any Possible x value
2. $\sum_{\text{all } x} f(x) = 1$

Cumulative Distribution Function (CDF)

Def: If x is a discrete r.V. It's CDF is given by: $F(x) = P(X \leq x)$

Gives the probability x will take a value less than or equal to.

EX

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{3} & \text{for } x \leq 3 \\ \frac{2}{3} & \text{for } x \leq 5 \\ 1 & \text{if } x \geq 6 \end{cases}$$

The Probability mass function

Is the function that maps out the possibilities

EX

x:	0	1	2
$f(x)$ $(P(X = x))$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

(Adds to 1)

Frequency Distribution for Observed Data

Observed Data - data one has seen

Frequency - how many times that number has occurred

Types of Frequency

Absolute Freq - Actual number of times a value occurs

Relative Freq - $\dfrac{\text{Absolute Freq}}{\text{Total \# of Data pts}}$

Pr. 2.5 b)

$2^4 = 16$ Possible outcomes

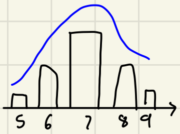
Pmf

x:	0	1	2	3	4
$f(x)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Binomial Distribution

Pr 2.6 a)

5	9
6	4 7
7	1 4 6 8
8	2 5
9	2

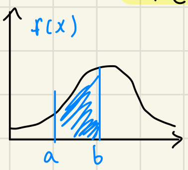


Norma Distribution of frequencies

Continous R.V, PDF and CDF.

Def. For a continous random variable X, the PDF $f(x)$ gives the Probability that X falls within a certain range. The probability that x lies between a and b is:

$$P(a \le X \le b) = \int_a^b f(x)\,dx \quad \Big\} \text{ PDF}$$



area under the graph.

Note: ① If x is cont r.v then $P(X=x)=0$

② The total area under the pdf curve is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Continuity Paradox

Def: Including or Excluding the endpoints does not matter since the prob of a single point is 0.

$$P(a < X \le b) = P(a \le x \le b) = P(a \le X < b) = P(a < x < b)$$

Problem 2.7

a) $f(x) = \begin{cases} K e^{-3x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_0^{\infty} K e^{-3x}\, dx = 1$$

$$K \int_0^{\infty} e^{-3x}\, dx = K \cdot \frac{e^{-3x}}{-3} \Big|_0^{\infty}$$

$$= K(0 - (-\tfrac{1}{3})) = \frac{K}{3} = 1$$

$\therefore K = 3$

b) Finding $P(0.5 \le x \le 1) = \int_{0.5}^{1} 3e^{-3x}\, dx$

$$\int_{0.5}^{1} 3e^{-3x}\, dx = 3\int_{0.5}^{1} e^{-3x} = -3 \cdot \frac{e^{-3x}}{3} \Big|_{0.5}^{1}$$

$= 0.173$

c) $P(X > 1) = \int_1^{\infty} 3e^{-3x}\, dx$

$P(X \le 0.5) = \int_0^{0.5} 3e^{-3x}\, dx$

The Cumulative Distribution Function for Continous R.V

Def: The CDF for continous R.V X, F(x) gives the probability X takes a value less than or equal to x

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)(t)\, dt, \text{ for } -\infty < x < \infty$$

Relationship between CDF and PDF

- The CDF is derived by integrating the PDF
- The PDF is found by differentiating the CDF

$$f_X(x) = \frac{d}{dx} F_X(x)$$

- $F(-\infty) = 0$, $F(\infty) = 1$

Problem 2.8

$f(x) = \begin{cases} 3e^{-3x}, & \text{if } x > 0 \\ 0, & \text{elsewhere} \end{cases}$

a) The CDF F(x) of X is found by integrating the PDF over x.

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)(t) \cdot dt$$

1. For $x \le 0$:

$F_X = 0$ for $x \le 0$

2. For $x > 0$:

$$F(x) = \int_0^{x} 3e^{-3t}\, dt$$

$$F(x) = -e^{-3t}\Big|_{t=0}^{x} = 1 - e^{-3x}$$

b)
1. $F(-\infty) = 0$
2. $F(\infty) = 1$

as $X \to \infty$, the term $e^{-3x} \to 0$

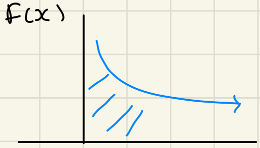
$F(\infty) = 1 - 0 = 1$

3. The CDF is non decreasing because the derivative of the CDF (PDF) is always non negative

$\therefore F_X(x)$ is non decreasing for all x

Want to Revisit?

$\therefore$ The CDF is $F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-3x} & x > 0 \end{cases}$



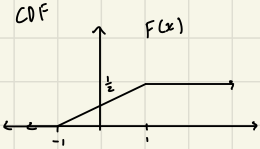
Problem 2.9

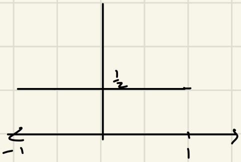
$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

If $x < -1$, C

If $x \ge 1$, x

CDF





$f(x) = \frac{d\,F(x)}{dx}$

To get P.D.F, we differentiate CDF

Note: $\int_0^1 5 \to 5x + C$

Problem 2.10

$$f(x) = \begin{cases} -x & -1 < x < 0 \\ x & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Integrate to find CDF

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ & \text{if } -1 < x < 0 \end{cases}$$

For $-1 < x < 0$

$\int_{-1}^{x} -t\,dt$

$= -\frac{t^2}{2}\Big|_{1}^{x} = \frac{1}{2} - \frac{x^2}{2}$

For $0 < x < 1$

$\int_{-1}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

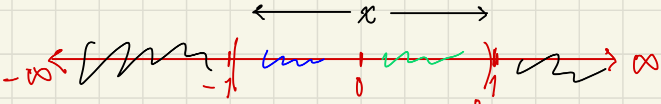
$\frac{1}{2} + \left(\frac{t^2}{2}\Big|_{0}^{x}\right)$

$= \frac{1}{2} + \frac{x^2}{2}$

$$F(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{2} - \frac{x^2}{2} & \text{if } -1 \le x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

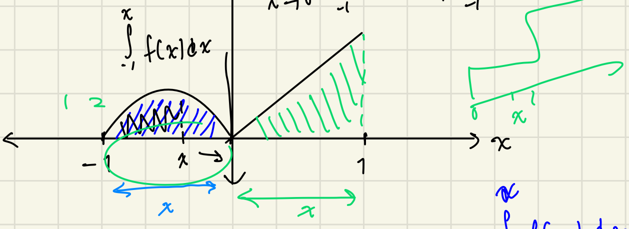
Note: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

Integrate up to x, not bounds.



$\int_{-1}^{x} f(x)\,dx \overset{??}{=} \int_{-1}^{0} f(x)\,dx$

$\lim_{x \to 0} \int_{-1}^{x} f(x)\,dx$, $\int_{-1}^{0} f(x)\,dx = c$



$\int_{-1}^{x} f(x)\,dx = c$

$\int$

Mathematical Expectations

Expected Value: The long run average value of a R.V, if you were to do an experiment many times.

For Discrete RV: Sum all possible outcomes, weighted by probability

$$E(X) = \sum_x x \cdot P(X=x)$$

For continous RV: Integrate the values, weighted by densities

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

Problem 2.11

a)

x:	1	-1
$f(x)$:	$1-\left(\frac{5}{6}\right)^4$ 0.518	$\left(\frac{5}{6}\right)^4$

$$P(X=-1) = \left(\frac{5}{6}\right)^4$$

$$E(X) = 1 \cdot 0.518 + (-1)\left(\frac{5}{6}\right)^4$$

$$= 0.036 \qquad > 0$$

$\therefore$ So he is expected to win a bit.

Properties of Expectations

1. $E(b) = b$, The expectation of a constant is simply the constant

2. $E(aX + bY) = aE(X) + bE(Y)$

If X and Y are r.v and a, b are constants, you can split them up.

3. $E(X+Y) = E(X) + E(Y)$

The expected value of a sum is sum of expected values

Problem 2.13

J, Q = \$15
K, A = \$5
2,3,4,5,6,7,8,9,10 = −\$4

X = money he makes from 1 draw

x:	15	5	-4
$f(x)$:	$\frac{8}{52}$	$\frac{8}{52}$	$\frac{36}{52}$

$$E(X) = 15 \cdot \frac{8}{52} + 5 \cdot \frac{8}{52} + (-4)\frac{36}{52}$$

$$E(x) = \$0.31$$

Note: This is a discrete random variable, this means he is expected to win 31¢ on avg.

Problem 2.14

$$f(x) = \begin{cases} \dfrac{4}{\pi(1+x^2)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)}\,dx \longrightarrow \text{how to integrate}$$

$$= \frac{2}{\pi}\ln(1+x^2)\Big|_{x=0}^{1}$$

$$= \frac{\ln 4}{\pi}$$

$$= 0.4413$$

$u = 1+x^2$

$du = 2x$

$$\int \frac{2\,du}{\pi \cdot u}$$

$$= \frac{2}{\pi}\ln u$$

Theorem 4.1

- Expectation of a function of a r.V
  If x is a discrete r.V with pmf $f(x)$,
  $$E[g(x)] = \sum_{\text{all } x} g(x) \cdot f(x)$$
  So, we apply some function g to X before taking Expectation
- If X is a con't r.V with pdf $f(x)$,
  then $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f(x) \cdot dx$

Problem 2.15

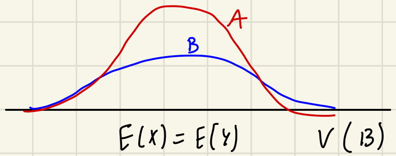
$$f(x) = \begin{cases} 2x, & \text{if } 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $E[X] = \int_0^1 x \cdot 2x \, dx = \frac{2x^3}{3}\Big|_0^1$
$= \frac{2}{3}$

b) $Y = 200X^{\frac{1}{2}} + 60$
$E[Y(X)] = \int_0^1 (200X^{\frac{1}{2}} + 60) \cdot 2x\,dx$
$= \$220$

c) $P(Y > 180) = P(200x^{\frac{1}{2}} + 60 > 180)$
$= P(200x^{\frac{1}{2}} > 120)$
$= P\left(x^{\frac{1}{2}} > \frac{120}{200}\right)$
$= P(X > (0.6)^2)$

$= \int_{0.36}^{1} 2x\,dx = x^2\Big|_{0.36}^{1}$
$= 0.8724$



Moments of a Random Variable

- a moment is a way to describe various aspects of a distribution of r.V X.
- The r'th moment of X is $\mu'_r = E(X^r)$
- Moments are used to understand shape and characteristics of distributions.

Ex

First Moment $(r=1)$: $E(X)$, the mean of X.
Second Moment $(r=2)$. $E(X^2)$

Central Moments:

The r-th central moment is $\mu_r = E[(X-\mu)^r]$
also called r'th moment about the mean.

Variance and Standard Deviation

The Variance is a measure of how much X deviates from it's mean μ.
This is the averaged square distance of X from it's mean.

- $Var(X) = E[(X-\mu)^2]$
$V(X) = \sigma^2$
$\sigma = \sqrt{V(X)}$: is called standard deviation of X. Denoted by σ.

Theorem 4.6

$$V(X) = E(X^2) - \boxed{[E(X)]^2}$$

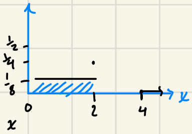
μ is a constant

$$= E[(X-\mu)^2]$$
$$= E[x^2 - 2\mu x + \mu^2]$$
$$= E(x^2) - E[2\mu x] + E(\mu)^2$$
$$= E(x^2) - 2\mu E(X) + \mu^2$$
$$= E(x^2) - 2\mu^2 + \mu^2$$
$$= E(x^2) - \mu^2$$

Problem 2.17

a) $f(x) = \begin{cases} \frac{1}{8} & \text{if } 0 \le x < 2 \\ \frac{x}{8} & \text{if } 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$

Note: $f(x) = \frac{2}{8} = \frac{1}{4}$ and @ $x=4$ $= \frac{1}{2}$



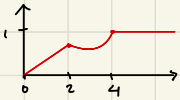
CDF $F(x)$

To find CDF, we integrate

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \frac{1}{8}\,dx = \frac{x}{8} & \text{if } 0 \le x < 2 \\ \int_0^2 \frac{1}{8}\,dx + \int_2^x \frac{t}{8}\,dt = \frac{1}{4} + \frac{x^2}{16}\Big|_2^x = \frac{x^2}{16} & \text{if } 2 \le x < 4 \\ 1 & \text{if } x \ge 4 \end{cases}$$

Is it continous?

Yes, it is.



$x=2,$
$F(x) = \frac{1}{4}$

To be continous:

- Non negative
- Adds up to 1

Variance: $V(X) = E[(X-\mu)^2]$

Standard Deviation: $= \sigma = \sqrt{V(X)}$

$$V(X) = E(X^2) - [E(X)]^2$$

Problem 2.16

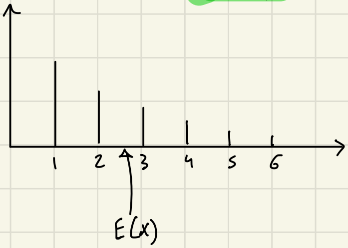
Smaller #: 1

X = min of the pair of #'s

Pmf

x:	1	2	3	4	5	6
$f(x)$:	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + \ldots + 6 \cdot \frac{1}{36}$$
$$= 2.54$$



$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum_{\text{all } x} x^2 \cdot f(x) = 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + \ldots 6^2 \cdot \frac{1}{36}$$
$$= 8.36$$

$$V(X) = 8.36 - (2.54)^2 = 1.91$$

Problem 2.17 b)

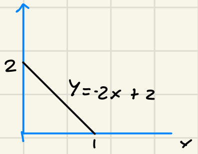
Expectations and Variance

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$
$$= \int_0^2 x \cdot \frac{1}{8}\,dx + \int_2^4 x \cdot \frac{x}{8}\,dx$$
$$= \frac{x^2}{8 \cdot 2}\Big|_0^2 + \frac{x^3}{3 \cdot 8}\Big|_2^4 =$$
$$\frac{4}{16} + \left[\frac{8}{3} - \frac{1}{3}\right]$$
$$E(x) = \frac{31}{12}$$

$$V(x) = E(x^2) - [E(x)]^2$$
$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$
$$= \int_0^2 x^2 \cdot \frac{1}{8}\,dx + \int_2^4 x^2 \cdot \frac{x}{8}\,dx$$
$$= \frac{47}{6}$$
$$V(x) = \frac{47}{6} - \left(\frac{31}{12}\right)^2$$
$$V(x) = 1.16$$

Problem 2.18

$$f(x) = \begin{cases} 2(1-x) & , \text{ if } 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$



a) Show $\mu'_r = \frac{2}{(r+1)(r+2)}$

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx = \int_0^1 x^r \cdot 2(1-x)dx$$

$$= \int_0^1 2x^r(1-x)dx = 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right] = \frac{2}{(r+1)(r+2)}$$

b) Using the Eq. $= \frac{2}{(r+1)(r+2)}$

$$V(x) = E(x^2) - [E(x)]^2$$

$$V(x) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

Notes:

1.) $V(X) \geq 0$

2.) $E(X^2) \geq [E(x)]^2$

3.) $V(aX+b) = a^2 \cdot V(X)$

for any constant a, b

Problem 2.19

$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{(x-3)}{2}} & , \text{ if } x > 3 \\ 0 & \text{otherwise} \end{cases}$$

a) $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)dx$

Let $u = x - 3$
$x = u + 3$
$dx = du$

$$E(x) = \int_3^x x \cdot \frac{1}{2}e^{\frac{-(x-3)}{2}}$$

$$E(x) = \frac{3}{2}\int_0^\infty ue^{-\frac{3}{2}}du + \frac{3}{2}\int_0^\infty 3e^{-\frac{3}{2}u}du$$

$$E(x) = \frac{3}{2}\left(\frac{8}{9} + 3 \cdot \frac{2}{3}\right) = 5$$

$$E(x^2) = \int_3^\infty x^2 f(x)dx = \int_3^\infty x^2 \cdot \frac{3}{2}e^{-\frac{3}{2}(x-3)}dx$$

$$E(x^2) = 29$$

$$var(x) = 29 - 5^2 = 4$$

b) $P(1 < X < 9) = 1 - e^{-3} = 0.95$

Chebyshev's Theorem

Provides a way to estimate the probability that a random variable lies within a certain number of standard deviations from the mean.

For any random variable X with mean μ and variance σ^2, for any $k > 1$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

or

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

k is the number of std's away it is from mean.

Note: This theorem provides a lower bound for the prob that X takes values within k std of the mean

Problem 2.20

a) $E(X) = 5$

$V(X) = 4 \rightarrow \sigma^2 = 4 \rightarrow \sigma = 2$

$P(1 < x < 9) = ? \longrightarrow P(-4 \le x - 5 \le 4)$

$P(|x - \mu| < k\sigma) \ge 1 - \frac{1}{k^2}$

$P(1 < x < 9) = P(|x-5| < 4)$ Equal

$P(|x-5| < 4) = 1 - P(|x-5| \ge 4)$

Since $\sigma = 2$, we set $k\sigma = 4 \therefore k = 2$

$P(|x-5| \ge 4) \le \frac{1}{k^2} = \frac{1}{4}$

$P(|x-5| < 4) = 1 - \frac{1}{4} = 0.75$

$k = \frac{L - \mu}{\sigma}$

$k = \frac{9-5}{2}$

$k = 2$

$1 - \frac{1}{k^2}$

$1 - \frac{1}{4} = 0.75$

b) $P(x \le 2 \text{ or } x \ge 8)$

$= P(x - 5 \le -3 \text{ or } x - 5 \ge 3)$

$= P(|x-5| \ge 3)$

Set $k\sigma = 3$, $k = \frac{3}{2} = 1.5$

$P(|x-5| \ge 3) \le \frac{1}{k^2} = \frac{1}{(1.5)^2} = \frac{1}{2.25}$

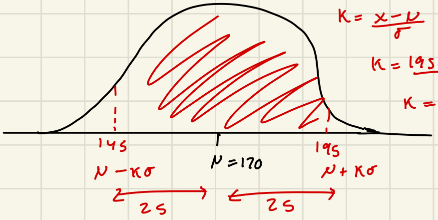
$P(x \le 2 \text{ or } x \ge 8) = 0.44$

$\frac{8-5}{2} = \frac{3}{2}$

$k = \frac{3}{2}$

$1 - \frac{1}{(1.5)^2}$

$= 0.44$



$k = \frac{x - \mu}{\sigma}$

$k = \frac{195 - 170}{10}$

$k = 2.5$

$1 - \frac{1}{k^2} = 1 - \frac{1}{2.5^2} = 0.84$

Notes on Chebyshev's Thm

- Applicable on any type of distribution
- Giveaway to use it, is the two bounds given are equal distant apart

Moment Generating Function

Recall that $\mu_r' = E(x^r)$, $r = 1, 2, 3$

If two random variables X and Y have the same r^{th} moments for all $r = 1, 2, 3$ then X and Y have the same probability distribution.

For a random variable X with pdf $f(x)$ the mgf is defined as $M_X(t) = E[e^{tx}] = \sum_{\text{all } x} e^{tx} \cdot f(x)$

For Continous R.V:

$$M_x(T) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

The moment generating function allows us to find all the different moments ($E[x^2]$, $E[x^{53}]$ etc)

Finding the first derivative of the MGF and setting it to 0 gives you First Moment

Second Derivative Set to 0, gives the Second moment.

First Moment: $\left.\frac{d\,m(t)}{dt}\right|_{t=0} = E(x)$

second moment $\left.\frac{d^2 m(t)}{dt^2}\right|_{t=0} = E(x^2)$

In general, $\left.\frac{d^r \cdot m(t)}{dt^r}\right|_{t=0} = E(x^r) = \mu_r'$

---

problem 2.21

$$f(x) = \frac{\binom{5}{x}\binom{3}{2-x}}{\binom{8}{2}} \quad \text{for } x = 0, 1, 2$$

1) Find pmf, by plugging in x values

2) Multiply e^{tx} times the probability of x's.

a) $M_X(t) = E(e^{tx}) = \sum e^{tx} \cdot f(x)$

Pmf:

x :	0	1	2
$f(x)$:	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

$= e^{t\cdot 0} \cdot f(0) + e^{t\cdot 1} \cdot f(1) + e^{t\cdot 2} \cdot f(2)$

$= \frac{3}{28} + \frac{15}{28} \cdot e^t + \frac{5}{14} e^{2t}$

b) Finding $E(X)$ and $V(X)$ through the mgf.

set $d\,m(t) = 0$

$E(x) = \left.\frac{d\,m(t)}{dt}\right|_{t=0}$ $= \frac{15}{28} e^t + \frac{5}{14} \cdot 2e^{2t}$

$= \frac{15}{28} + \frac{10}{14} = \frac{35}{28} = \frac{5}{4}$

$E(x) = \frac{5}{4}$

Second derivative:

$= \left.\frac{15}{28} e^t + 4e^{2t} \cdot \frac{5}{14}\right|_{t=0}$

$= \frac{15}{28} + \frac{10}{7}$

$E(x^2) = \frac{55}{28}$

$V(x) = E(x^2) - [E(x)]^2$

$V(x) = \frac{55}{28} - (\frac{5}{4})^2$

$V(x) = \frac{45}{112}$

Problem 2.22

$$f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) For a continous r.V

$$m_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{tx} \cdot e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{(t-\frac{1}{2})x}$$

$$= \frac{1}{2} \left. \frac{e^{(t-\frac{1}{2})x}}{2t-1} \right|_{x=0}^{\infty}$$

The mgf must be a finite value

$$= \lim_{x \to \infty} \frac{e^{(t-\frac{1}{2})x}}{2t-1} - \frac{1}{2t-1}$$

$$m(t) = \begin{cases} \frac{1}{2t-1} - \frac{1}{2t-1} = 0 & \text{if } t = \frac{1}{2} \\ -\frac{1}{2t-1} = \frac{1}{1-2t} & \text{if } t < \frac{1}{2} \\ \infty & \text{if } t > \frac{1}{2} \end{cases}$$

The m.g.f of X is $m(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$.

b) $\frac{d\,m(t)}{dt} = \left.\frac{2}{(1-2t)^2}\right|_0$ $\quad \frac{dm^2(t)}{dt^2} = \frac{6}{(1-2t)^3}$

$E(x) = 2$

$E(x^2) = 8$

$V(X) = E(x^2) - [E(x)^2]$

$V(x) = 8 - 2^2$

$V(x) = 4$

If two r.vs have the same m.g.f they must have the same distribution.

Problem 2.23

$$m(t) = \frac{1}{6}\boxed{\ } + \frac{2}{6} e^{\boxed{2}t} + \frac{3}{6} e^{\boxed{3}t}$$

$$M_X(t) = E\left[e^{tx}\right]$$

The random variables take on the values of 0, 2, 3 because these are the coeffecient on the t terms.

The terms suggest : $P(X=0) = \frac{1}{6}$

$P(X=2) = \frac{2}{6}$

$P(X=3) = \frac{3}{6}$

The Pmf gives You probabilities directly

The mgf repesents these probabilities in a transformed way.

Topic 3

Topic 3

• Bernoulli Distribution and Binomial Distribution

Definition: A bernoulli trial is an experiment with 2 mutually exclusive distinct outcomes. There are Successe's (P) and Failures (1-P).

Example: Toss a coin, Shoot target...etc

Let $x =$ # of success in a bernoulli trial

$$x = \begin{cases} 1, & \text{if success} \\ 0 & \text{failure} \end{cases}$$

$P(x=1)=\theta$ $\qquad$ $p(x=0)=1-\theta$

Probability Mass Function For Bernoulli random variable X

$$P(X=x)=p^{x}(1-p)^{1-x}, \text{ where } x=0,1$$

Way to write: $X \sim \text{Bernoulli}(\theta)$

Binomial Distribution

- A binomial experiment consists of n identical and independent Bernoulli trials

Binomial Random Variable: $x \sim \text{Binomial}(n, k)$

PMF of Binomial Distribution

For a binomial random variable X, the probability of getting exactly K successes in n trials is $P(X=k) = \binom{n}{k} p^{k}(1-p)^{n-k}$.

Binomial Distribution

1. Consists of n identical trials
2. independent trial
3. Each trial is a Bernoulli trial
4. Prob of success is θ in each trial

Mean: np $\qquad$ var: $np(1-p)$

Problem 3.1

$\frac{2}{5}$ have O +

$\frac{1}{15}$ have O−

3 randomly selected

X # of donors with O+ blood

Recognize, $x \sim Bin(n=3, \theta = \frac{2}{5})$

n is # of trials, θ is probability of success.

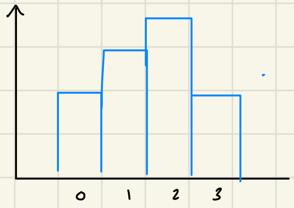
Values that x can take on, X: 0, 1, 2, 3.

Find PMF: $f(0) = P(X=0) = \left(1 - \frac{2}{5}\right)^3$

$P(X=1) = \binom{3}{1} \cdot \left(\frac{2}{5}\right)\left(1 - \frac{2}{5}\right)^2$

$P(x=2) = \binom{3}{2}\left(\frac{2}{5}\right)\left(1 - \frac{2}{5}\right)$

$P(x=3) = \left(\frac{2}{5}\right)^3$



Using the Pmf of binomial dist. $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$. $x = 0, 1, 2, 3$

$f(x) = \binom{3}{x}\left(\frac{2}{5}\right)^x\left(1 - \frac{2}{5}\right)^{3-x},$

Note: Bernoulli(θ) is a special case of Bin(n, θ) with $n=1$.

Thm 5.4

The mgf of $x \sim Bin$ is $m(t) = [1 + \theta(e^t - 1)]^n$

Proof:

$$M(t) = E(e^{tx})$$
$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x}\theta^x(1-\theta)^{n-x}$$
$$= \sum_{x=0}^{n}\binom{n}{x}(\theta e^t)^x(1-\theta)^{n-x}$$
$$= [1 + \theta(e^t - 1)]^n$$

Thm

If $x \sim Bin(n, \theta)$, then $E(x) = n\theta$, $V(x) = n\theta(1-\theta)$

Pf: $\left.\frac{dm(t)}{dt}\right|_{t=0} = n\theta$

$$\frac{d^2 m(t)}{dt^2} = n(n-1)[1 + \theta(e^t - 1)]^{n-2}\theta e^t \cdot \theta e^t + n[1 + \theta(e^t - 1)]^{n-1}\theta e^t$$

$$E(x^2) = n(n-1)\theta^2 + n\theta$$

$$V(x) = n(\theta - \theta^2) = n\theta(1-\theta)$$

Problem 3.2

Recovery rate 0.3

15 People selected (n trials)

a) Let X be the # of People among the 15 who recovered.

$x \sim \text{Bin}(n=15, \theta=0.3)$

a) $P(x \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.7^{15} + \binom{15}{1} \cdot 0.3^1 \cdot 0.7^{14} + \dots \binom{15}{3} 0.3^3 \cdot 0.7^{15-3}$

$= 0.297$

b) $P(X \geq 2) = P(X=2) + \dots + P(X=15)$

$1 - P(X=0) - P(X=1) = 0.965$

Geometric Distribution

The geometric distribution is a discrete Probability distribution that models the number of trials needed to achieve the first Success

One parameter is θ, where θ is Probability of Success in each trial.

PMF: $P(X=x) = (1-p)^{x-1} \cdot p$

The first x-1 trials must be failures times the Probability of Success. P is Prob of Success

$E[X] = \frac{1}{\theta}$ $\quad Var(X) = \frac{1-\theta}{\theta^2}$

Distribution	Models
Bernouli	Single trial
Binomial	Number of successes in n trials
Geometric	Number of trials to first Success.

Problem 3.3

a) Let x = # of trials untill 6 appears

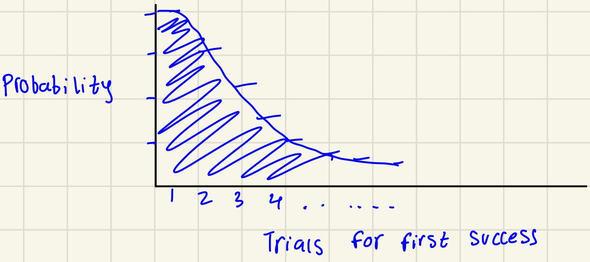
$x \sim \text{Geo}(\theta = \frac{1}{6})$

$P(x=3) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) = \frac{25}{216}$

If its a 6 on the third toss, had to have been 1-5 on first two.

b) $P(X \geq 3) = 1 - P(X=1) - P(X=2)$

$= 1 - \frac{1}{6} - \left(\frac{5}{6}\right) \cdot \frac{1}{6} = \frac{25}{36}$



The first trial always has highest Probability at $X=1$.

$P(X \leq 3) = P(X=1) + P(X=2) + P(X=3)$

$= (1-0.3)^{1-1} \cdot 0.3 + (1-0.3)^1 \cdot 0.3 + (1-0.3)^2 \cdot 0.3$

$= 0.657$

For Geometric Distribution:

Can also use the CDF: $F(x) = P(X \leq x) = 1 - (1-p)^x$

Types of Distributions

Formula:

Bernouli → Models a single mutually exclusive independent trial → $\underbrace{p}_{\text{success}}{}^{x}\underbrace{(1-p)}_{\text{failures}}{}^{1-x}$ x is # of successes

Binomial → Follows n Bernoulli trials, counts the number of successes. → $\binom{n}{k} p^{k} (1-p)^{n-k}$

K successes in n trials

Bernoulli distribution is nothing but the Binomial distribution, where the trial only occurs once.

Binomial → $E(x) = n\underbrace{p}_{\text{wins}}$, $Var(x) = \underbrace{n \cdot p}_{\text{wins}}\underbrace{(1-p)}_{\text{Losses}}$

Geometric → Number of trials to first success → $\underbrace{(1-p)^{x-1}}_{\text{Failures}} \cdot \underbrace{p}_{\text{success}}$

Negative Binomial → Number of trials it takes to get certain # of successes $\binom{k-1}{r-1} \cdot p^{r} (1-p)^{k-r}$

r is # of success

K is amt of trials

Hypergeometric → Drawing from finite population, without replacement $\dfrac{\overbrace{\binom{M}{x}}^{\text{ways to choose}}\binom{N-M}{n-x}}{\binom{N}{n}}$ ← ways to choose the "rest" → Total outcomes

Binomial questions have a set number of trials "out of n trials" "in n trials"

Negative Binomial questions have a set number of successes. "number of successess"

Problem 3.4

The memoryless property of the geometric distribution, is a unique characteristic where probability of needing more trials for a success, independent from how many have been completed

a) Show that for a positive integer a,

$$P(X>a) = q^a, \text{ where } q = 1-p$$

Failure : $q = 1-p$

Event $X>a$ means that the first a trials are all failures.

$$P(X>a) = (1-p)^a = q^a$$

b) $P(X>a+b \mid X>a) = P(X>b)$

By the Definition of Conditional Probability:

$$P(X>a+b \mid X>a) = \frac{P(X>a+b \cap X>a)}{P(X>a)}$$

Because $P(X>a+b)$ implies $P(X>a)$ the event can just be

$$= \frac{P(X>a+b)}{P(X>a)}$$

From a) $P(X>a) = q^n$ and $P(X>a+b) = q^{a+b}$

$$= \frac{q^{a+b}}{q^a} = q^b = P(X>b)$$

Negative Binomial Distribution

Models the number of trials needed to achieve a specified number of successes in a sequence of independent and identically distributed Bernoulli trials.

Formula:

If X reperesents the number of trials needed to get r successes, X follows negative Binomial distribution

$$\text{PMF} = P(X=k) = \binom{k-1}{r-1} \cdot p^r (1-p)^{k-r}$$

- p is the probability of success
- 1-p prob of failure
- r is # of required successes
- k is the total number of trials, needed for r success
- $\binom{k-1}{r-1}$ Binomial Coeff, counts ways to arrange r-1 success in k-1 trials.

Expected value: $E[X] = \frac{r}{p}$

Variance: $\text{Var}(X) = \frac{r(1-p)}{p^2}$

Cool Note:

When $r=1$, negative binomial distribution becomes geometric

The first $x-1$ trials must result in $r-1$ successes.

Problem 3.5

a) $P = 0.3$
$r = 2$
$X = 10$

$$P(X=K) = \binom{K-1}{r-1} \cdot p^{r} (1-p)^{K-r}$$

$$P(X=10) = \binom{10-1}{2-1} \cdot 0.3^{2} (1-0.3)^{8}$$

$$P(X=10) = \binom{9}{1} \cdot 0.3^{2} (0.7)^{8}$$

$$P(X=10) = 0.0467$$

b) $P(X \leq 4) = P(X=2) + P(X=3) + P(X=4)$

$$= 0.3^{2} + \binom{3-1}{2-1} \cdot 0.3^{2} \cdot \boxed{0.7^{1}} + \binom{4-1}{2-1} \cdot 0.3^{2} \cdot \boxed{0.7^{2}}$$

$$= 0.348$$

This term is introduced because we have 1 and then 2 failures within the first x trials.

Where $r=2$, $X =$ Changes, $P=0.3$

Problem 3.6

$P = 0.5$ $F = 0.5$
$X = 4, 5, 6, 7$
$r = 4$

4 wins means series is over

Probability: one team wins 4 in a row is $P(\text{sweep}) = 0.5^{4}$
$P(\text{Sweep}) = \frac{1}{16}$

Prob of 5 games: $P(X=5) = \binom{4}{3} \cdot 0.5^{4} \cdot 0.5 = 4 \cdot \frac{1}{32} = \frac{1}{8}$

Prob of 6 games: $P(X=6) = \binom{5}{3} \cdot 0.5^{4} \cdot 0.5^{2} = 10 \cdot \frac{1}{64} = \frac{10}{64} = \frac{5}{32}$

Prob of 7 games: $P(X=7) = \binom{6}{3} \cdot 0.5^{4} \cdot 0.5^{3} = 20 \cdot \frac{1}{128} = \frac{5}{32}$

Pmf Let Y be # of games till one team wins.

$y =$	4	5	6	7
$f(y) =$	0.125	0.25	0.3125	0.3125

Multiply all Probabilities by 2.

Problem 3.7

Series length	4	5	6	7
Obsered Freq	17	15	16	12
Observed prob:	$\frac{17}{60}$	$\frac{15}{60}$	$\frac{16}{60}$	$\frac{12}{60}$
	$= 0.283$	$= 0.25$	$= 0.267$	$= 0.20$

No, they dont fit a similiar distribution, teams ussually are not evenly matched up in the finals.

Hypergeometric Distribution

Probability distribution that describes the likelihood of getting a certain number of successes in a sequence of draws from a finite population without replacement.

Without Replacement - Impies that probabilities will change based on draws, making the draws dependent.

Key Characteristics

- Finite population, divided into successes and failures
- Draw x items without replacement.
- Drawing specific number of success in given amt of tries

$$\text{PMF: } P(X=M) = \frac{\overbrace{\binom{M}{x}}^{\text{ways to choose}} \binom{N-M}{n-x}}{\binom{N}{n}}$$

← ways to choose the "rest"

→ Total outcomes

Where:

- N is total size of population
- M is total "Successess" in population
- n is the number of draws
- X is random var reperesenting the # of success that we want.

Mean: $\mu = \frac{nM}{N}$

$$\text{Var}(x) = \frac{nM(N-M)(N-n)}{N^2(N-1)}$$

The term $\frac{N-n}{N-1}$ is called the finite population adjustment.

Note: If sampling less than 5% of the total population, the hypergeometric and binomial distribution will give "close" answers.

Problem 3.8

With Replacement - Independent
Without replacement - Not Independent

Box contains : 8 red balls
2 black balls

a) Hypergeometric:

$$P(4\text{ Red}) = \frac{\binom{8}{4}\cdot\binom{2}{1}}{\binom{10}{5}} = \frac{5}{9}$$

b) $P(R) = 0.8$
$P(B) = 0.2$
Total outcomes:

$$\text{Bin} \sim \underbrace{\binom{5}{4}}_{\text{Choosing 4/5 Red}} \cdot \underbrace{(0.8)^4}_{\text{Prob of choosing reds}} \cdot \underbrace{0.2}_{\text{leftover from the 1 black}}$$

works with geo and neg binomial

* c) $f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$

$p = 0.8$
$x = 4$
$k = 1$

$$f(x=4) = \binom{4-1}{1-1} \cdot 0.8^1 (1-0.8)^{4-1}$$
$$= \binom{3}{0} \cdot 0.8^1 (0.2)^3$$
$$= \frac{4}{625}$$

Problem 3.9

120 applicants
80 qualified
40 not qualified

a) $$\frac{\binom{80}{2}\cdot\binom{40}{3}}{\binom{120}{5}} = 0.164$$

b) $p = \frac{2}{3}$

$$P(x=2) = \binom{5}{2}\left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^3 = \frac{40}{243} = 0.1646$$

Bernoulli - single trial
Binomial - independent bernoui trials
Geometric - till first success
Neg Binomial - # of trials, needed for # of success.
Hypergeometric - finite population without replacement

Derivatives Reminders

Product rule: $f(x)\cdot g(x) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$

Quotient : $\frac{f(x)}{g(x)} = \frac{f'(x)\cdot g(x) \cdot g'(x)\cdot f(x)}{g(x)^2}$

Exponential : $e^u = e^u \cdot u'$

Lawns : $a^u = a^u \cdot u' \cdot \ln a$ $\quad 7^{2x-5} = 7^{2x-5} \cdot 2 \cdot \ln 7$

Constants: $\frac{c}{f(x)} = c \frac{dx}{dy} f(x)'$

: $\frac{f(x)}{c}$: $\frac{f'(x)}{c}$

Practise Midterm #2 - Test 2

1) a) Let x reperesent a student born in november in grade 2.

20 students

$P(\text{Nov}) = \frac{1}{12}$

$P(x \le 2) = ?$

This can be modelled using Binomial

$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$ $\quad x \sim \text{Bin}(n=20, p=\frac{1}{12})$

$P(x=0) = \left(\frac{11}{12}\right)^{20} = 0.17548$

$P(x=1) = \binom{20}{1} \cdot \left(\frac{1}{12}\right) \cdot \left(\frac{11}{12}\right)^{19} = 0.319$

$P(x=2) = \binom{20}{2} \cdot \left(\frac{1}{12}\right)^2 \cdot \left(\frac{11}{12}\right)^{18} = 0.2755$

$P(x \le 2) = \left(\frac{11}{12}\right)^{20} + \left(\frac{1}{12}\right)\left(\frac{11}{12}\right)^{19} + \left(\frac{1}{12}\right)^2 \left(\frac{11}{12}\right)^{18}$

$P(x \le 2) = 0.770$

Key words

Set number of n ("in 10 games") means $\text{Bin}(n=10, p= \ )$

$$\binom{n}{k}(p)^k(1-p)^{n-k}$$

Set number of successes means negative Binomial

b) 5 students until he finds 2 born in nov

Modelled by binomial Distribution.

$Y \sim \text{Binomial}(n=4, p=\frac{1}{12})$

$P(Y<2) = P(Y=0) + P(Y=1)$

$P(Y=0) = \binom{4}{0} \cdot \left(\frac{1}{12}\right)^0 \cdot \left(\frac{11}{12}\right)^4$

$P(Y=1) = \binom{4}{1} \cdot \left(\frac{1}{12}\right)^1 \cdot \left(\frac{11}{12}\right)^3$

$P(Y<2) = 0.706 + 0.25675$

$= 0.963$

2) $$f(x) = \begin{cases} \frac{2}{5} & \text{if } 0 < x \le 1 \\ 3/5 & \text{if } 1 < x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

a) To find the probability $P(0.5 \le X \le 1.5)$ we integrate $f(x)$

$$P(a \le x \le b) = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} \frac{2}{5}\,dx + \int_{1}^{1.5} \frac{3}{5}\,dx$$

$$\frac{2}{5}x \Big|_{0.5}^{1} + \frac{3}{5}x \Big|_{1}^{1.5}$$

$= \frac{1}{5} + \frac{3}{10} = 0.5$ $\quad P(0.5 \le X \le 1.5) = 0.5$

Note: From density function, to get probabilities, you integrate over that region.

From CDF, $P(a \le x \le b) = F(b) - F(a)$

2b) $$f(x) = \begin{cases} \frac{2}{5} & \text{if } 0 < x \le 1 \\ 3/5 & \text{if } 1 < x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_0^x \frac{2}{5}\,dx = \frac{2}{5}x \Big|_0^x = \frac{2}{5}x$$

$$\int_1^x \frac{3}{5}\,dt = \frac{3}{5}t \Big|_1^x = \frac{3}{5}x - \frac{3}{5} = \frac{3}{5}(x-1)$$

Combine: $\frac{2}{5} + \frac{3}{5}(x-1)$

$= \frac{2}{5} + \frac{3}{5}x - \frac{3}{5}$

$= \frac{3}{5}x - \frac{1}{5}$

$$F(X) = \begin{cases} 0, & \text{if } 0 \le x \\ \frac{2}{5}x, & \text{if } 0 < x \le 1 \\ \frac{3}{5}x - \frac{1}{5} & \text{if } 1 < x \le 2 \\ 1, & \text{if } x > 2 \end{cases}$$

Solving CDF Questions

Note: To find CDF from PDF, set up bounds to x and remember it is cumulative, so you must add the previous intervals probability, evaluated at the start of the new bounds.

c) $E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

$E[X] = \int_0^1 x \cdot \frac{2}{5}\,dx = \frac{2}{2\cdot 5}x^2\Big|_0^1 = \frac{1}{5}$

$= \int_1^2 \frac{3}{5}x\,dx = \frac{3}{2\cdot 5}x^2\Big|_1^2 = \frac{9}{10}$

$= \frac{1}{5} + \frac{9}{10} = 1.1$

$E[x] = 1.1$

$E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx$

$= \int_0^1 \frac{2}{5}x^2\,dx + \int_1^2 \frac{3}{5}x^2\,dx$

$= \frac{2}{15}x^3\Big|_0^1 + \frac{3}{15}x^3\Big|_1^2$

$= \frac{2}{15} + \frac{7}{5} = \frac{23}{15}$

$E[x^2] = \frac{23}{15}$

$\text{Var}(x) = E[x^2] - (E[x])^2$

$\text{Var}(x) = \left(\frac{23}{15}\right) - (1.1)^2$

$\text{Var}(x) = \frac{97}{300}$

3)

$$f(x) = \begin{cases} e^{-(x-1)} & \text{if } x \geq 1 \\ 0 & \text{, otherwise} \end{cases}$$

$e^{tx} \cdot e^{-x} \cdot e$

$(e^{t} \cdot e^{-1})^x \cdot e$

$(e^{t-1})^x \cdot e$

$e^{(t-1)x+1}$

$M_x(t) = E[e^{tx}]$

$\int_1^\infty e^{tx} \cdot e^{-(x-1)}\,dx = e^{tx-x+1} = e^{(t-1)}$

$M_x(t) = \int_1^\infty e^{(t-1)x+1}\,dx = e\int_1^\infty e^{(t-1)x}\,dx = \frac{e^{(t-1)x}}{(t-1)}$

$= \frac{e^{(t-1)x}}{t-1}\Big|_1^\infty$

As $x \to \infty$, if $t < 1$, then $e^{(t-1)x} \to 0$

when $x=1$, $\frac{e^{t-1}}{t-1} = \frac{0 - e^{t-1}}{t-1} = \frac{-e^{t}}{t-1}$

$M_x(t) = e \cdot \frac{-e^{t-1}}{t-1} = \frac{e^{t}}{t-1}$ for $t < 1$

b) $\frac{dm}{dt} = \frac{e^t}{t-1}\Big|_0$

$= \frac{(e^t)(t-1) - (t)(e^t)}{(t-1)^2}\Big|_0$

$E(x) = 2$

Key Properties for Exponentials

- For e^{ax}, if $a > 0$, e^{ax} grows toward infinity as $x \to \infty$
  if $a < 0$, e^{ax} decays toward 0 as $x \to \infty$

Behavior at infinity:

$a < 0$, $e^{ax} \to 0$ as $x \to \infty$

$a > 0$, $e^{ax} \to \infty$ as $x \to \infty$

Note: Given a PMF and asked for CDF, is simply the probabilities adding on to each other.

To Find Mean/Expected value: Solve the integrals of the PDF.

How to Solve MGF Questions

1) Determine if You are given a continous r.V or discrete r.V. (continous means PDF, Discreate means PMF). Then use either $\sum_{K}^{\infty} e^{tx}\cdot f(x)$ or $\int_{-\infty}^{\infty} e^{tx}\cdot f(x)$.

2) Factor exponential terms/ simplify
3) Solve integral
4) Look at behaviour to see how, we can make f(x) a finite value.
5) Plug in limit, Solve
6) Find First/Second moment from derivatives.

Exponent Rules

1) $e^{a}\cdot e^{b} = e^{a+b}$

2) $e^{a\cdot bx} = e^{(ab)x}$

$(e^{2x}\cdot e^{-3x}) = e^{(2-3)x} = e^{-x}$

---

1) MGF practise Questions

$$\int_0^{\infty} e^{tx}\cdot 3e^{-3x}\,dx$$

$$3\int_0^{\infty} e^{tx}\cdot e^{-3x} = 3\int_0^{\infty} e^{tx-3x} = e^{x(t-3)}$$

$$\int_0^{\infty} e^{x(t-3)} = \frac{e^{x(t-3)}}{(t-3)}\Bigg|_0^{\infty}$$

To make $e^{x(t-3)}$ approach 0, t must be less than 3.

$x(t-3) \leq 0$

$t-3 \leq 0$

$t \leq 3$

$$= 0 - \frac{1}{(t-3)}$$

$$= -\frac{3}{t-3}$$ is the mgf?

2) $P(X=0) = 0.5$ Find mgf:

$P(X=1) = 0.3$

$P(X=2) = 0.2$

1) Discrete: $M_X(T) = \sum_{K}^{\infty} e^{tx}\cdot f(x)$

$$= e^{0}\cdot 0.5 + e^{t}\cdot 0.3 + e^{2t}\cdot 0.2$$

$$= 0.5 + 0.3e^{t} + 0.2e^{2t}$$

2) Find $E[x]$: Derive, set to 0.

$$\frac{d}{dt} = 0.3e^{t} + 0.2e^{2t}$$

$$= 0.3e^{t} + 0.4e^{2t}\Big|_0$$

$$= 0.3 + 0.4 = 0.7$$

3) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Finding mgf.

Continous R.V: $M_X(T) = \int_0^{\infty} e^{tx}\cdot \frac{1}{2}e^{-\frac{x}{2}}\,dx$

$$= \frac{1}{2}\int_0^{\infty} e^{tx}\cdot e^{-\frac{x}{2}}\,dx = \frac{1}{2}\int_0^{\infty} e^{(tx-\frac{x}{2})}\,dx$$

$$= \frac{1}{2}\cdot\int_0^{\infty} e^{x(t-\frac{1}{2})} = \frac{1}{2}\cdot\frac{e^{x(t-\frac{1}{2})}}{t-\frac{1}{2}} = \frac{e^{x(t-\frac{1}{2})}}{2t-1}\Bigg|_0^{\infty}$$

for $\frac{e^{x(t-\frac{1}{2})}}{2t-1}$ to approach 0, t must be less than $\frac{1}{2}$ $\therefore$ $t < \frac{1}{2}$

$$0 - \frac{1}{2t-1} = \frac{-1}{2t-1}$$

$$\sum_{K=0}^{\infty} ar^{K} = \frac{a}{1-r} \text{ for } |r|<1$$

$$\frac{-1}{2t-1} = (0)(2t-1) - (-1)(2)$$

$$= \frac{2}{(2t-1)^2}\Bigg|_0 = E[x] = 2$$

3) $P(X=K) = \frac{1}{3}\left(\frac{2}{3}\right)^{K}$, $K=0,1,2\ldots$

1) Discrete r.V implies we use this def $M_X(T) = \sum_{K}^{\infty} e^{tx}\cdot f(x)dx$ because values of K are infinite we do:

$$\sum_{K}^{\infty} e^{tx}\cdot \frac{1}{3}\left(\frac{2}{3}\right)^{K} dx = \frac{1}{3}\cdot\sum_{K}^{\infty} e^{tK}\cdot\left(\frac{2}{3}\right)^{K} = \left(\frac{2e^{t}}{3}\right)^{K}$$

$$= \frac{1}{3}\sum_{K=0}^{\infty}\left(\frac{2e^{t}}{3}\right)^{K} \text{ where } r = \frac{2e^{t}}{3},\ a = 1$$

a = 1 because K=0 gives 1

Because $|r| < 1$, Setup inequality $\frac{2e^{t}}{3} < 1$

$$\frac{2e^{t}}{3} < 1 = 2e^{t} < 3 = e^{t} < \frac{3}{2} = t < \ln\frac{3}{2}$$

$$= \frac{1}{3}\cdot\frac{1}{1-\left(\frac{2}{3}e^{t}\right)} = \frac{1}{3-2e^{t}} \text{ for } t < \ln\frac{3}{2}$$

$$\frac{d}{dt} = \frac{(0)(3-2e^{t}) - (1)(-2e^{t})}{(3-2e^{t})^2}\Bigg|_0 = \frac{2e^{t}}{(3-2e^{t})^2}\Bigg|_0$$

$E[x] = 2$

1) $P(X = x) = \frac{1}{2^x}$, $x = 1, x=2, x=3$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ for $|r|<1$

$M_x(T) = \sum_{x=1}^{\infty} e^{tx}.f(x)$

$= \sum_{x=1}^{\infty} e^{tx}\cdot \frac{1}{2^x} = \sum_{K}^{\infty} \left(\frac{e^t}{2}\right)^K$

$r = \frac{e^t}{2}$ $a = \frac{e^t}{2}$

$\left(\frac{e^t}{2}\right)^K$ so $t<0$

$= \frac{\frac{e^t}{2}}{1 - \frac{e^t}{2}}$ for $t < 0$

$= \frac{e^t}{2-e^t}$

Binomial: $\binom{n}{k} p^k (1-p)^{n-k}$

More MGF practise

Problem 2.22

$$f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{, otherwise} \end{cases}$$

a) This is a continous r.v

$\therefore$ we use $M_{Tx} = \int_{-\infty}^{\infty} e^{tx}. f(x)\, dx$

$\int_0^{\infty} e^{tx} \cdot \frac{1}{2} e^{-\frac{x}{2}}$

$\frac{1}{2}\int_0^{\infty} e^{tx}\cdot e^{-\frac{x}{2}} = \frac{1}{2}\int_0^{\infty} e^{tx - \frac{x}{2}} = e^{x(t-\frac{1}{2})}$

$\frac{1}{2}\int_0^{\infty} e^{x(t-\frac{1}{2})}$ Need $t - \frac{1}{2} < 0$

$t < \frac{1}{2}$

$= \frac{1}{2} \cdot \frac{e^{x(t-\frac{1}{2})}}{t-\frac{1}{2}}\Big|_0^{\infty}$ For $t < \frac{1}{2}$

$\to \frac{e^{x(t-\frac{1}{2})}}{2t-1}\Big|_0^{\infty}$ $\lim_{x\to\infty}$ $= -\frac{1}{2t-1}$

First moment

$= \frac{-1}{2t-1} \Big|$ Set derivative to 0

$f' = \frac{2}{(2t-1)^2}$ $E[X] = 2$

$f'' = \frac{2\cdot 2\cdot 2}{(2t-1)^3}$

$E(x^2) = 8$

$V[x] = 8 - 2^2 = 4$

$m(t) = \frac{1}{6} + \frac{2}{6} e^{2t} + \frac{3}{6} e^{3t}$

Def: $M_T = \sum e^{tx}.f(x)$

In this case $x = 1, 2, 3$

The pmf is

$x=1$	$\frac{1}{6}$
$x=2$	$\frac{2}{6}$
$x=3$	$\frac{3}{6}$

Practise Midterm #2 - Test 1

1) Independent - Binomial

a) Alice wins 7 times in no more than 10 games.

There are 36 outcomes:
There are 6 ways to tie : $\frac{1}{6}$
Splitting the remaining 30 outcomes both ways gives us $P(\text{win}) = \frac{15}{36} = \frac{5}{12}$

$$f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$$

$$P(X \leq 10) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \left(\frac{5}{12}\right)^7 + \binom{7}{6}\left(\frac{5}{12}\right)^7\left(\frac{7}{12}\right) + \binom{8}{7}\left(\frac{5}{12}\right)^7\left(\frac{7}{12}\right)^2 + \binom{10-1}{7-1}\left(\frac{5}{12}\right)^7\left(1-\frac{5}{12}\right)^3$$

$= 0.068$ ∴ is the prob she wins 7 times in no more than 10 games

$P(\text{win}) = \frac{5}{12}$

1b) Binomial :

$$P(X \geq 8) = P(x=8) + P(x=9) + P(x=10)$$

$$= \binom{8}{8}\cdot\left(\frac{5}{12}\right)^8\cdot\left(\frac{7}{12}\right)^0 + \binom{9}{8}\cdot\left(\frac{5}{12}\right)^8\cdot\left(\frac{7}{12}\right)^1 + \binom{10}{8}\cdot\left(\frac{5}{12}\right)^8\cdot\left(\frac{7}{12}\right)^2$$

$= 0.019588 \times 2 = 0.03917$

Binomial has fixed n

Negative Binomial has r successes fixed.

1c) Alice wins $\frac{5}{7}$
Draws $\frac{2}{7}$

Bob wins first time in less than 10 games.

Geo $\sim (p = \frac{5}{12})$

$$f(x) = \frac{5}{12}\left(\frac{7}{12}\right)^0 + \frac{5}{12}\left(\frac{7}{12}\right)^1$$

$= 0.660$

2)
$$f(x) = \begin{cases} e^{2x} & \text{for } x < 0 \\ 1-x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{, otherwise} \end{cases}$$

a) 1) For $x < 0$

$$\int_{-\infty}^{x} e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_{-\infty}^{x} = \frac{1}{2}e^{2x} - 0 = \frac{1}{2}e^{2x}$$

2) For $0 \leq x \leq 1$

Plug in $0 = \frac{1}{2}$

$$\int_0^x 1-x\,dx = x - \frac{x^2}{2}\Big|_0^x = x - \frac{x^2}{2} + \frac{1}{2}$$

3) For $x \geq 0$ Plug in $1 := 1$

$$\int_1^x f(x) = 0 + 1 = 1$$

CDF:
$$\begin{cases} \frac{1}{2}e^{2x} & \text{for } x < 0 \\ x - \frac{x^2}{2} + \frac{1}{2} & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

How to sketch this function?

2b) Let $Y = -6X + 3$, Find $P(-1.5 \leq Y < 6)$

1) Sub in : $P(-1.5 \leq -6x+3 \leq 6)$
2) Subtract 3 from both sides: $P(-4.5 \leq -6x \leq 3)$
3) divide both sides by 6 : $P(\frac{3}{4} \leq x \leq -\frac{1}{2})$
∴ $P(-\frac{1}{2} \leq x \leq \frac{3}{4})$

Using CDF: $F(\frac{3}{4}) - F(-\frac{1}{2})$
$= \frac{31}{32} - 0.19393$
$= 0.785$

2c) Given $Y = -6X + 3$

$$E[aX+b] = aE[X] + b$$
$$E[-6X+3] = -6E[X] + 3$$

$$E[X] = \int_{-\infty}^{0} x\cdot e^{2x}\,dx + x\int_0^1 x(1-x)\,dx$$

$x - x^2$
$\frac{x^2}{2} - \frac{x^3}{3}\Big|^1$
$= \frac{1}{2} - \frac{1}{3}$
$= \frac{1}{6}$

$$\int_{-\infty}^{0} x\cdot e^{2x}$$

$u = x$
$du = 1$
$dx = du$
$v = \frac{1}{2}e^{2x}$
$dv = e^{2x}$

$\int u\,dv = uv - \int v\,du$
$= x\cdot\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$
$= x\cdot\frac{1}{2}e^{2x} - \frac{1}{2} + \frac{e^{2x}}{2}\Big|_{-\infty}^{0}$
$= 0 - \frac{1}{2}$ ∴ $-\frac{1}{2} + \frac{1}{6}$
$= -\frac{1}{3}$

$-6 \cdot -\frac{1}{3} + 3$
$= 5$

3) $f(x) = 3\left(\frac{1}{4}\right)^{x+1}$, where $x=0,1,2$

a) $M_x(T) = E\left(e^{tx}\right)$

1) Find PMF :

$$M(t) = \sum_{x=0}^{\infty} e^{tx} \cdot f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot 3\left(\frac{1}{4}\right)^{x+1} \checkmark$$

$$= 3 \cdot \frac{1}{4} \sum_{x=0}^{\infty} \left(e^{t} \cdot \frac{1}{4}\right)^{x}$$

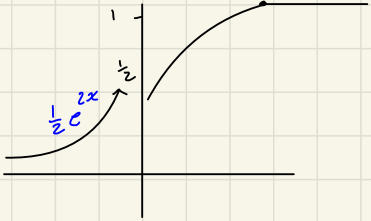
$$= \frac{3}{4} \sum_{x=0}^{\infty} \left(e^{t} \cdot \frac{1}{4}\right)^{x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{4}\right)^{x} = \frac{1}{1-\frac{e^t}{4}}$$

$$= \frac{3}{4} \cdot \frac{4}{4-e^t} = \frac{3}{4-e^t}$$

for $t < \ln 4$

2 a) sketch



3b) $E[x] = \frac{3}{4-e^t}$

$$= \left.\frac{-(3)(-e^t)}{(4-e^t)^2}\right|$$

$$= \frac{3e^t}{4-e^t} = \frac{3}{9} = \frac{1}{3}$$

$$E[x^2] = \frac{3e^t}{(e^t-4)^2} \quad \frac{d}{dt} = \frac{(3e^t)(e^t-4)-(e^t)(3e^t)}{(e^t-4)^4}$$

$$E[x^2] = -\frac{3e^t(e^t+4)}{(e^t-4)^3} \quad @\, t=0 \quad = \frac{5}{9}$$

$\text{Var}(x) = E[x^2] - (E[x])^2$

$$= \frac{5}{9} - \left(\frac{1}{3}\right)^2 = \frac{4}{9}$$

---

8 questions, 3 answers to each

a) Exactly 4 correct answers.

$P(\text{right answer}) = \frac{1}{3}$

8 Total question

$P(x=4)$, Independent trials

r successes means neg Binomial dist

$$f(x) = \binom{8}{4}\left(\frac{1}{3}\right)^4\left(1-\frac{1}{3}\right)^{8-4}$$

$$= \frac{1120}{6561} = 0.17$$

$$P(x \geq 2) = 1 - P(x=1) + P(x=0)$$

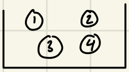
$$\binom{8}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^7 + \binom{8}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^8$$

$$= 1 - 0.156 - 0.039$$

$$= 0.805$$

Assignment 3 Questions

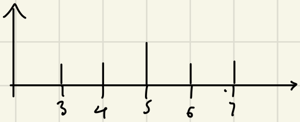
1)



a) Numbers on the 2 balls

	Z values
(1, 2)	3
(1, 3)	4
(1, 4)	5
(2, 3)	5
(2, 4)	6
(3, 4)	7

Z :	3	4	5	6	7
f(z)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$



b)

$$F(z) = \begin{cases} 0, & z < 3 \\ \frac{1}{6}, & 3 \leq z < 4 \\ \frac{2}{6}, & 4 \leq z \leq 5 \\ \frac{2}{3}, & 5 < z \leq 6 \\ \frac{5}{6}, & 6 \leq z < 7 \\ 1, & z \geq 7. \end{cases}$$

2) a) $P(x \leq 3) = P(x=2) + P(x=1) + P(x=0)$

$P(x \leq 3) =$

$$P(-0.4 < x < 4) = F(4) - F(-0.4)$$
$$= \frac{3}{4} - \frac{1}{4}$$
$$= \frac{1}{2}$$

$P(x=5) = \frac{1}{4}$ $\quad P(x \geq 1) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(x \leq 3) = \frac{3}{4}$

$P(x < 3) = \frac{1}{2}$

Given the CDF:

1) To find $P(X \leq a)$, $P(X \leq 3)$, simply just find $F(3)$

2) For $P(X > a)$, $P(X > 5)$, simpy use $1 - P(X \leq 5) = 1 - F(5)$

$f(x) = 2\left(\frac{1}{3}\right)^x$ for $x = 1, 2, 3 \ldots$

1) This is a discrete r.V

$$M_x(T) = \sum_x^{\infty} e^{tx} \cdot f(x)$$
$$= \sum_x^{\infty} e^{tx} \cdot 2\left(\frac{1}{3}\right)^x$$
$$= 2 \cdot \sum_x^{\infty} e^{tx} \cdot \left(\frac{1}{3}\right)^x$$
$$= 2 \cdot \sum_x^{\infty} \left(\frac{1}{3}e^t\right)^x \text{ where } r = \left(\frac{1}{3}e^t\right),\ a = \frac{1}{3}e^t$$

$\longrightarrow$ to ensure $|r| < 1$, $3 \times \frac{1}{3}e^t < 1 \times 3$

$e^t < 3$

$t < \ln 3$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \text{ for } |r| < 1$$

$$= 2 \cdot \frac{\frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{\frac{2}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{2e^t}{3 - e^t}$$

$$\frac{d}{dt} = \frac{(2e^t)(3-e^t) - (-e^t)(2e^t)}{(3-e^t)^2} = \frac{2e^t(3-e^t) + (e^t)(2e^t)}{(3-e^t)^2}$$

$6e^t - 2e^{2t} + 2e^{2t}$

$= \frac{6e^t}{(3-e^t)^2}$

$\mu = 0.260$

$\sigma = 0.005$

$K = \frac{Lim - \mu}{\sigma}$

$1 - \frac{1}{k^2} = \frac{35}{36}$

$-\frac{1}{k^2} = \frac{35}{36} - 1$

$-1 \times \frac{-1}{k^2} = \frac{-1}{36} \times -1$

$\frac{1}{k^2} = \frac{1}{36}$

$k^2 = 36$

$k = 6$

$6 = \frac{Lim - 0.260}{0.005}$

$\frac{3}{100} = Lim - 0.260$

$\frac{3}{100} + 0.260 = Lim$

$Lim_1 = 0.29$

This is upper lim

$\therefore$ Lower < 0.23

CDF Related Questions

1) given the pdf:

$$f(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq 1 \\ \frac{1}{2} & 1 < x \leq 2 \\ \frac{1}{2}(3-x) & 2 \leq x \leq 3 \\ 0 & 0 \text{ therwise} \end{cases}$$

$$F(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(x)\,dx$$

1) when $x < 0$

$$\int_{-\infty}^{0} f(x) = 0$$

2) when $0 \leq x \leq 1$

$$\int_{0}^{x} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^x$$

$$= \frac{x^2}{4}$$

3) when $1 < x \leq 2$

$$\int_{1}^{x} \frac{1}{2}\,dx = \frac{1}{2}x\Big|_1^x$$

$$= \left(\frac{1}{2}x - \frac{1}{2}\right) + \frac{1}{4}$$

$$= \frac{1}{2}x - \frac{1}{4}$$

4) when $2 \leq x \leq 3$

$$\int_{2}^{x} \frac{1}{2}(3-x)\,dx = \frac{3x}{2} - \frac{x^2}{4}\Big|_2^x$$

$$= \left(\frac{3x}{2} - \frac{x^2}{4} - 2 + \frac{3}{4}\right)$$

$$= \frac{3}{4}$$

$$= \left(\frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}\right)$$

5) when $x > 3$

$$\int_{3}^{\infty} f(x) = 0$$

$$\therefore\ f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{4} & \text{if } 0 \leq x \leq 1 \\ \frac{1}{2}x - \frac{1}{4} & \text{if } 1 < x \leq 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} & \text{if } 2 \leq x \leq 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

2a) $f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

case 1: $-\infty$ to 0

$$\int_{-\infty}^{x} f(x)\,dx = 0$$

case 2: $0 \leq x \leq 1$

$$\int_{0}^{x} 2x\,dx = \frac{2x^2}{2}\Big|_0^x = \frac{2x^2}{2} + 0 = \frac{2x^2}{2} = x^2$$

case 3: $x \geq 1$ plug 1: 1

$$\int_{1}^{\infty} x^2\,dx = \frac{x^3}{3}\Big|_1^{\infty} = \infty - \frac{1}{3} = \infty + 1 = \infty$$

$$F(x): \begin{cases} 0 & X \leq 0 \\ x^2 & 0 \leq X \leq 1 \\ 1 & X \geq 0 \end{cases}$$

2b) $P(0.25 \leq x \leq 0.75)$

Using CDF, we see

$$P(0.25 \leq x \leq 0.75) = F(0.75) - F(0.25)$$

$$= \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$$

2c) $E[X] = \int x \cdot f(x)$ on the range of X

$$E[X] = \int_0^1 x \cdot 2x\,dx$$

$$= \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3} - 0$$

$$E[X] = \frac{2}{3}$$

3) $f(x) = \begin{cases} 4xe^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

a) case 1: $x \leq 0$

$$\int_{-\infty}^{0} f(x) = 0$$

case 2: $x \geq 0$

$$\int_0^x 4xe^{-2x}\,dx$$

$$\int u\,dv = uv - \int v\,du$$

BP

$U = 4x$

$du = 4$

$dv = e^{-2x}$

$v = \frac{e^{-2x}}{-2}$

✓ $$\int_0^x 4x \cdot e^{-2x} = 4x \cdot \frac{e^{-2x}}{2} - \int_0^x \frac{e^{-2x}}{-2} \cdot 4\,dx$$

$$= -2xe^{-2x} + 2 \cdot -\frac{1}{2}e^{-2x}$$

$$= -2xe^{-2x} - e^{-2x} = e^{-2x}(-2x + 1)$$

$$=$$

b) $P(X \leq 1) = F_X(1) = 1 - e^{-2}(2+1)$

$$= 1 - 3e^{-2}$$

More CDF

1) $f(x) = \begin{cases} \frac{1}{4}x & 0 \le x < 2 \\ \frac{1}{8} & 2 \le x < 4 \\ 0 & \text{0therwise} \end{cases}$ Find $P(1 < x \le 3)$

$$\int_{-\infty}^{0} f(x) = 0 \quad \text{From } -\infty \text{ to } 0$$

$$\int_{0}^{x} \frac{1}{4}x\,dx = \frac{1}{4 \cdot 2}x^2 = \frac{1}{8}x^2 \Big|_0^x = \frac{1}{8}x^2 + 0 = \frac{1}{8}x^2$$

$$\int_{2}^{x} \frac{1}{8}dx = \frac{1}{8}x \Big|_2^x = \frac{1}{8}x - \frac{1}{4} + \frac{1}{2} = \frac{1}{8}x + \frac{1}{4}$$

$$\int_{4}^{\infty} f(x) = 1$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{8}x^2 & 0 \le x < 2 \\ \frac{1}{8}x + \frac{1}{4} & 2 \le x < 4 \\ 1 & x > 4 \end{cases}$$

$$P(1 \le x \le 3) = F(3) - F(1)$$

$$P(1 \le x \le 3) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$$

2) $f(x) = \begin{cases} \frac{3}{2}x^2 & 0 \le x \le 1 \\ 2 - x & 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{0} f(x)\,dx = 0 \quad \text{For } x < 0$$

$$\int_{0}^{x} \frac{3}{2}x^2\,dx = \frac{1}{2}x^3 \Big|_0^x = \frac{1}{2}x^3 - 0 + 0 = \frac{1}{2}x^3$$

$$\int_{1}^{x} 2 - x\,dx = 2x - \frac{x^2}{2} \Big|_1^x = 2x - \frac{x^2}{2} - \left(2 - \frac{1}{2}\right) + \frac{1}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

$$\int_{2}^{\infty} f(x) = 1$$

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{2}x^3 & \text{for } 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & \text{for } 1 \le x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$$

To Find the Expected value in CDF Questions:

1) use PDF integrated $\int_{-\infty}^{\infty} x \cdot f(x)$ and add up all valid intervals for that function. $\rightarrow E[X]$

2) Then do the Same for $E[X^2]$

Poisson Distribution

The Poisson distribution models the number of rare events that occur in a fixed interval of time or space.

The events must be:

1.) Randomly and Independently of each other
2.) At a constant average rate (λ) (number of expected outcomes)
3.) Events can not occur simultaneously

PMF for Poisson Distribution

$$P(X=K) = \frac{\lambda^{K} \cdot e^{-\lambda}}{K!}, \quad K = 0, 1, 2$$

where:

X = number of events in interval
λ = average rate of events.
e = 2.718...

Note: The Poisson distribution is a limiting case of the binomial distribution.

- $n \to \infty$
- $p \to 0$
- np remains constant

When n is large and p is small
Rule $n \geq 20$, $p \leq 0.05$

Note:

$E[X] = \lambda$
$\mathrm{var}(X) = \lambda$

The mean and variance are equal to lambda.

If lambda is closer to 0, Poisson is Right skewed. If lambda is infinity, it is symmetric.

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

Calculator Tip

- Poisson PD for exact: $P(X=4)$, $\lambda = 5$
- Poisson CD for $X \leq K$: $P(X \leq 1) = P(X=0) + P(X=1)$

$$P(X \geq 4) = 1 - P(X \leq 3)$$

Problem 3.10

$\lambda = 5$

a) $P(X < 3) = P(x=0) + P(X=1) + P(X=2)$

$P(X<3) = 0.0006738 + 0.0337 + 0.08422$

$P(X<3) = 0.12466$

b) $P(X \geq 4)$ in exactly 3 of the next 5 days

$P(X \geq 4) = 1 - P(X \leq 3)$

$= 1 - 0.265$

$= 0.735 \longrightarrow$ Probability of success

Fixed 3 days from 5.

$\text{Bin}(n=5,\ P=0.735)$

$= \binom{5}{3} (0.735)^3 (1-0.735)^2$

$= 0.279$

Problem 3.11

$P = 0.00005$

$n = 10\,000$

$x = 2$

a) $P(X=0) + P(X=1) + P(X=2)$

$(0.00005) + \binom{10\,000}{1} (0.00005)^1 (1-0.0005)^1 + \binom{10\,000}{2} (0.0005)^2$

$P = 0.9856$

b) Approximate with $nP = 0.00005 \times 10\,000$

$P = 0.9856$

Thm 5.9. The mgf of Pois(λ) is $m(t) = e^{\lambda(e^t - 1)}$

Pf: $m(t) = E(e^{tx})$

$$= \sum_{x=0}^{\infty} e^{tx} \underbrace{\left(\frac{\lambda^x \cdot e^{-\lambda}}{x!} \right)}_{\text{Pmf for Poisson}}$$

$$= \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x \cdot e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \underbrace{\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}}_{\text{In the form of geometric series}}$$

$$= e^{-\lambda} \left(\frac{\lambda e^{t\,0}}{0!} + \frac{\lambda e^{t\,1}}{1!} + \frac{\lambda e^{t\,2}}{2!} + \dots \right)$$

$$= e^{-\lambda}\left(e^{\lambda e^t}\right) = e^{\lambda(e^t - 1)}$$

Showing $E(X) = \lambda$ by mgf

$m'(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t$

$$E(X) = \left. \frac{dm(t)}{dt} \right|_{t=0} = \lambda$$

Problem 3.12

$\lambda = 2$

$X = 50 - 2Y - Y^2$

$E(X) = E(50 - 2Y - Y^2)$

$= 50 - 2E(Y) - E(Y)^2$

$\underline{E(Y) = \lambda = 2}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \underbrace{V(Y)}_{V(Y) = \lambda} + [E(Y)]^2$

$= 2 + 2^2 = 6$

$E(X) = 50 - 2 \cdot 2 - 6 = 40$

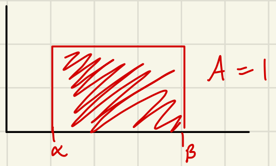
$\therefore$ Expected profit is \$40.

Topic 4

Uniform Distribution

The uniform distribution models situations where every outcome in a certain interval is equally likely.

General: A r.v X takes values in an interval $[\alpha, \beta]$ with all values being equally likely $x \sim \text{Uniform}(\alpha, \beta)$



Note: The expected value provides us no relevant information. The expected value and the mean gives us no predictive power.

For a Continous R.V distributed on $[a, b]$

PDF:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

- Height of PDF is constant because outcomes are equally likely.
- Area under curve is 1

CDF

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

Mean:

$$E(X) = \frac{a+b}{2} \quad \text{(average of intervals)}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Discrete Uniform Distribution

- Probability of each outcome:

$$P(X = x) = \frac{1}{n}, \quad x \in \{1, 2 \ldots n\}$$

- Mean: $E(X) = \frac{n+1}{2}$

$$\text{Var}(X) = \frac{n^2-1}{12}$$

Problem 4.1

Let x = waiting time in minutes untill the bus comes.

$x \sim \text{unif}(0, 5)$, 8:04, 8:05

$P(X \ge 3) = \frac{2}{5}$, because $\frac{1}{5}$ per min.

$$f(x) = \begin{cases} \frac{1}{5} & \text{if } 0 \le x \le 5 \\ 0 & \text{else} \end{cases}$$

$$P(X > 3) = \int_3^5 \frac{1}{5} dx = \frac{x}{5}\Big|_3^5 = \frac{2}{5} = 0.4$$

Thm 6.1

$x \sim \text{unif}(a,b)$, then $E(x) = \frac{a+b}{2}$ and $V(x) = \frac{1}{12}(b-a)^2$

PF:

$$E(X) = \int_a^b \frac{1}{b-a} x \, dx = \frac{x^2}{2(b-a)} \Big|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

$$E(x^2) = \int_a^b x^2 \frac{1}{b-a} \, dx = \frac{x^3}{3(b-a)} \Big|_a^b$$

$$= \frac{B^3-a^3}{3(B-a)} = \frac{a^2+ab+b^2}{3}$$

$$V(X) = E(x^2) - [E(x)]^2$$

$$= \frac{1}{3}(a^2+ab+b^2) - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{12}(a^2-2ab+b^2)$$

$$V(X) = \frac{(b-a)^2}{12}$$

The Exponential Distribution

Exponential Distribution describes the time between successive events in a poisson process.

- Events occur independently
- Average rate of occurrence is constant

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where λ is the rate parameter: number of events per unit time

$f(x)$ decays exponentialy as x increases.

CDF

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

If $x \sim Exp(\lambda)$

$$E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}$$

Relationship Poisson and Exponential Distribution

Poisson describes chances of event occuring

Exponential gives distance between events that are described by poisson situation.

Solving Problems with Exponential Distribution

1.) Find λ: $\lambda = \frac{\text{\# of events}}{\text{Total time}}$

2.) Use PDF for exact probabilities:

$$P(X = x),\ f(x) = \lambda e^{-\lambda x}$$

3.) Use CDF

To find $P(X \leq x)$: $F(x) = 1 - e^{-\lambda x}$

To find $P(X > x)$: Use $1 - F(x) = e^{-\lambda x}$

Problem 4.4: Volcano

36 observations of X

Average time between eruptions 36.72 months

or $\lambda = 0.027$ per month

Density Scaled Histogram

Time Interval	Observed Freq	Density
$0 \leq x < 20$	13	0.0181
$20 \leq x < 40$	9	0.0125
$40 \leq x < 60$	5	0.0069
⋮	⋮	⋮

$$\text{Density} = \frac{\text{Observed Freq}}{\text{Total \# of Observations} \times \text{Bin width}}$$

Total Area = 1

Memoryless Property

$$P(X > s + t \mid X > s) = P(X > t)$$

The probability of an event occurring in the future does not depend on how much time has already passed.

$$P(x > s + t \mid x > s) = \frac{P(x > s+t \cup x > s)}{P(x > s)}$$

$$= \frac{P(x > s+t)}{P(x > s)}$$

1.) $P(x > s) = \int_{s}^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx$

$= e^{-\frac{s}{\lambda}}$

2.) $P(x > s + t) = e^{-\frac{s+t}{\lambda}}$

Gamma Distribution

- The gamma distribution is Continous distribution, that generalizes the exponential distribution
- The gamma distribution handles waiting times for $r \geq 1$ events.

The Gamma Distribution is parameterized:

- Shape Parameter ($\alpha > 0$): Reperesents the number of events.
- Rate Parameter ($\beta > 0$): Reperesents the rate of process. (Scale)

PDF:

$$f(x; r, \lambda) = \frac{\lambda^r x^{r-1} \cdot e^{-\lambda x}}{\Gamma(r)}, \quad x > 0$$

where $\Gamma(r)$ is the gamma function, a generalization of factorial: $\Gamma(r) = \int_0^{\infty} t^{r-1} e^{-t} \, dt$

Key Properties:

Mean: $E[X] = \frac{r}{\lambda}$ or $\alpha\beta$

Variance: $Var[x] = \frac{r}{\lambda^2}$ or $\alpha\beta^2$

Special Cases:

when $r=1$, gamma dist reduces to exponential dist

Note: The gamma distribution models how long you'll wait for multiple events to occur. As r increases, it becomes more symmetric, resembling normal distribution.

Example

15 /hr

$P(X < 10) = 1 - P(X \geq 10)$ for first 3 customers to arrive.

15/60 min → 1 /4min → $\lambda = 4$

and the number of times we are interested in is $r = 3$.

$$P(x \leq 10) = \int_0^{10} \frac{x^{3-1} \cdot e^{-x/4}}{(3-1)!\, 4^3} = 0.456$$

Properties of the Gamma Function

i) $\Gamma(1) = 1$

This is the base case for the gamma function.

$$\Gamma(1) = \int_0^\infty y^{1-1} e^{-y}\, dy = \int_0^\infty e^{-y}\, dy$$

The integral of e^{-y} from 0 to ∞ is 1.

ii) $\Gamma(\alpha) = (\alpha - 1)\, \Gamma(\alpha - 1)$

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

$$= -y^{\alpha-1} \cdot e^{-y} \Big|_0^\infty + \int_0^\infty e^{-y} (\alpha-1)\, y^{\alpha-2}\, dy$$

$$= (\alpha-1) \cdot \int_0^\infty \cdot y^{\alpha-2} \cdot e^{-y}\, dy$$

$$\cdot\ (\alpha-1)\, \Gamma(\alpha-1)$$

iii) If n is a positive integer, $\alpha = n$

$\Gamma(n) = (n-1)!$

$$P(n) = (n-1) \cdot P(n-1) = (n-1)(n-2) \cdot P(n-2)$$

$$\therefore\ \Gamma(n) = (n-1)!$$

Problem 4.5)

$$f(x) = \begin{cases} k x^3 e^{-x/2}, & \text{if } K > 0 \\ 0, & \text{elsewhere} \end{cases}$$

a) To find K: $\int_0^\infty K x^3 e^{-\frac{x}{2}}\, dx = 1$

$x \sim (\alpha = 4, \beta = 2)$

normalizing constant

$$K = \frac{1}{\beta \cdot P(\alpha)} = \frac{1}{2^4 \cdot \Gamma(4)} = \frac{1}{2^4 \cdot 3!} = \frac{1}{96}$$

Problem 4.6

Asked to show $\mu_r = \beta^r \dfrac{\Gamma(\alpha + r)}{\Gamma(\alpha)}$

for $\alpha + r \geq 0$.

The r^{th} Moment of X is:

$$\mu_r = E[X^r] = \int_0^\infty x^r f_X(x)\, dx$$

Subbing in PDF

$$\mu_r = \int_0^\infty \underbrace{x^r}_{} \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} \underbrace{x^{\alpha-1}}_{} \cdot e^{\frac{-x}{\beta}}\, dx$$

Combine

Then use u substitution:

Let $u = x/\beta$, $x = \beta u$, $dx = \beta\, du$

$$\mu_r = \frac{1}{\beta^\alpha \Gamma(\alpha)} \beta^{r+\alpha} \int_0^\infty u^{r+\alpha-1} \cdot e^{-u}\, du$$

This is in the form of the gamma function

$$\Gamma(r + \alpha) = \int_0^\infty u^{r+\alpha-1} \cdot e^{-u}\, du$$

$$\mu_r = \frac{\beta^{r+\alpha}}{\beta^\alpha \Gamma(\alpha)} \Gamma(r + \alpha)$$

$$\mu_r = \beta^r \frac{\Gamma(r + \alpha)}{\Gamma(\alpha)}$$

$\therefore$ The r^{th} moment of

$x \sim \text{Gamma}(\alpha, \beta)$

is $\mu_r = \beta^r \dfrac{\Gamma(r + \alpha)}{\Gamma(\alpha)}$

Problem 4.7

$$f(y) = \begin{cases} ky(1-y)^2, & \text{if } 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) To find K, we know

$$\int_0^1 ky(1-y)^2\,dy = 1$$

$$= K\int_0^1 y(1-2y+y^2)\,dy$$

$$= K\left(\tfrac{1}{2} - 2\cdot\tfrac{1}{3} + \tfrac{1}{4}\right) = 1$$

$$K\cdot\tfrac{1}{12} = 1 \qquad \therefore K = 12$$

b) The Probability all counters are busy for more than half the day.

$$P(Y > 0.5) = \int_{0.5}^1 f(y)\,dy = \int_{0.5}^1 12y(1-y)^2\,dy$$

$$= \int_{0.5}^1 12y - 12y^3\,dy$$

$$= \frac{12y^2}{2} - \frac{12y^4}{4}\Big|_{0.5}^1$$

$$= \frac{5}{16}$$

c) Find Expected proportion of time all counters are busy.

$$E[Y] = \int_0^1 y(12y(1-y)^2dy$$

$$E[Y] = \int_0^1 12y^2(1-y)^2\,dy$$

$$E[Y] = \int_0^1 12y^2 - 24y^3 + 12y^4\,dy$$

$$= \frac{12y^3}{3} - \frac{24y^4}{4} + \frac{12y^5}{5}\Big|_0^1$$

$$E[Y] = \frac{2}{5}$$

Special Cases for Gamma

1) When Shape Parameter $\alpha = 1$, it becomes the exponential distribution

2) Chi-Square (X^2) distribution

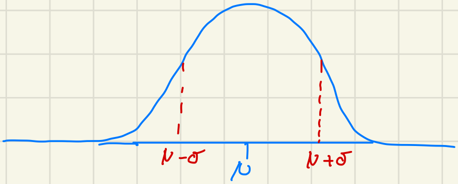
- This occurs when:
  - Shape Parameter ($\alpha = \frac{V}{2}$) where V is the degrees of freedom
  - Scale Parameter ($\beta = 2$)

In which case: $E[x] = V$

$V[x] = 2V$

The Normal Distribution

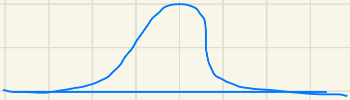
- This is a probability distribution that is symmetric about it's mean.
- The shape of it's PDF is bell curved



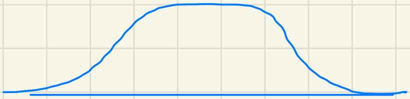
Key Characteristics

- Mean (μ): Determines centre of distribution
- Standard Deviation (σ): Determines how spread the curve is.

If σ is smaller:



If σ is larger:



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

The normal distribution is asymptotic and never actually touches 0.

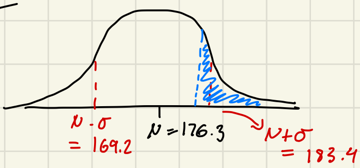
Emperical Rule

- 68% of data falls within 1 standard deviation ($\mu \pm \sigma$)
- 95% of data falls within 2 standard deviations ($\mu \pm 2\sigma$)
- 99.7% of data falls within 3 standard deviations ($\mu \pm 3\sigma$)

The Standard Normal Distribution

- Is when a normal distribution with $\mu=0$ and $\sigma^2=1$, $\sigma=1$

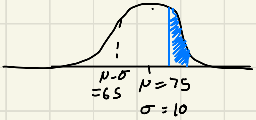
Ex



Normal CD: $P=0.3$

To find 90^{th} Percentile, first Standardize: $z = \frac{x-\mu}{\sigma}$

Problem 4.9



a) Find between 80 and 90

Normal CD:

$P=0.2417$

$x \sim N(\mu=75, \sigma^2=10^2)$

$$P(80 < x < 90) = P\left(\frac{80-75}{10} < \frac{x-75}{10} < \frac{90-75}{10}\right)$$

$$= P(0.5 < z < 1.5)$$

$$= 0.9332 - 0.6915$$

b) Find score x → 95 th percentile

$$P(X \leq x) = 0.95$$

$$P\left(z \leq \frac{x-75}{10}\right) = 0.95$$

$$\frac{x-75}{10} = 1.645$$

$$x = 75 + 10 \times 1.645 = 91.45$$

$$P(z > 1.645) = 0.05$$

$$z_{0.5} = 0$$

Thm 6.6

The mgf of a r.v X that follows a normal Distribution

$x \sim N(\mu, \sigma^2)$ is:

$$M_x(t) = E[e^{tx}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Proof:

$$M_x(t) = E[e^{tx}]$$

PDF: $F_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx \xrightarrow{} = x^2 - 2\mu x + \mu^2$$

Detour:

Complete the Square:

$$\frac{-x^2}{2\sigma^2} + \left(t + \frac{\mu}{\sigma^2}\right)x = -\frac{1}{2\sigma^2}\left(x^2 - 2x\sigma^2\left(t + \frac{\mu}{\sigma^2}\right)\right)$$

Factor/simplify $\frac{\sigma^2}{2}\left(t^2 + \frac{2t\mu}{\sigma^2} + \frac{\mu^2}{\sigma^4}\right) = \mu t + \frac{\sigma^2 t^2}{2}$

$$\int_{-\infty}^{\infty} e^{\frac{-(x-\sigma)^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2}$$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Show $E(X) = \mu$, $V(X) = \sigma^2$

PF: $\frac{dm(t)}{dt} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)$

$$E(X) = \left.\frac{dm(t)}{dt}\right|_{t=0} = \mu$$

$$\frac{d^2 m(t)}{dt^2} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2 t)^2 + e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot \sigma^2$$

$$E(x^2) = \left.\frac{d^2 m(t)}{dt^2}\right|_{t=0} = \mu^2 + \sigma^2$$

$$V(x) = E(x^2) - [E(x)]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Thm: If $x \sim N(\mu, \sigma^2)$, then

$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$

1) $M_Z(t) = E\left[e^{t\frac{X-\mu}{\sigma}}\right]$

2) $M_Z(T) = E\left[e^{\frac{t}{\sigma}X} e^{\frac{-t\mu}{\sigma}}\right]$

3) $M_Z(t) = e^{\frac{-t\mu}{\sigma}} \cdot M_X\left(\frac{t}{\sigma}\right)$

4) $M_Z(t) = e^{-\frac{t\mu}{\sigma}} \cdot e^{\mu\frac{t}{\sigma} + \frac{1}{2}\left(\frac{t}{\sigma}\right)^2 \sigma^2}$

$$= M_Z(t) = e^{\frac{1}{2}t^2}$$

Linear Transformation of Normal Distribution

Thm: $x \sim N(\mu, \sigma^2)$ and $Y = ax + b$ then:

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

Pf:

$$E[Y] = E[ax+b] = a\underbrace{E[x]}_{} + b = a\underbrace{\mu}_{} + b$$

$$Var(Y) = var(ax+b) = a^2 \underbrace{Var(x)}_{} = a^2\underbrace{\sigma^2}_{}$$

pretty much replacing definitions here

Normal Approximation to Binomial Distribution

For a binomial r.v $x \sim Bin(n, P)$ with:

$$\mu = nP$$
$$\sigma^2 = nP(1-P)$$

as $n \to \infty$, the distribution of x can be approximated by the normal distribution. $x \sim N(nP, nP(1-P))$

Pf

$$Z = \frac{x - E(x)}{\sqrt{var(x)}} = \frac{x - nP}{\sqrt{nP(1-P)}}$$

The mgf of Z converges to the mgf of the standard normal dist as $n \to \infty$

Conditions for Approximation

1)
- $nP > 5$
- $n(1-P) > 5$

2) Correction for Continuity:

Since the binomial dist is discrete and the normal dist is continous. Apply a continuity correction: $P(X \le K)$: $P(x \le K + 0.5)$

Example of Approximation

1) $x \sim Bin(n=25, \theta=0.4)$

Find $P(x=8)$.

$$P(x=8) = \binom{25}{8} 0.4^8 = 0.120$$

2) $x \approx (nP = 25 \cdot 0.4 = 10, n\theta(1-\theta) = 6)$

Let $Y \sim N(10, 6)$

$$P(Y = 8) = P(7.5 < Y \le 9.5)$$

$$= P(\underbrace{-1.02 < z -0.61}_{\text{StN Table}}) = 0.117$$

Topic 5

Joint, Marginal and Conditional Distributions

These are multivariate distributions, which study the behaviour of multiple random variables at the same time.

Joint Distributions

- X and Y are two discrete r.v's
- The joint PMF is:

$$f(x,y) = P(X=x, Y=y)$$

This gives the probability $X=x$ and $Y=y$ at the same time. (the full picture)

Properties of Joint Dist

1.) $f(x,y) \geq 0$

2.) $\sum_x \sum_y f(x,y) = 1$

Adding up all probabilities for every possible pair of (x,y)

Note: If X and Y are indep the joint PMF will factor into
$f(x,y) = f_X(x) \cdot f_Y(y)$

Marginal Distributions

- The marginal PMF describes the probability distribution of one of the r.V's on its own, ignoring the other var.

- To find marginal PMF of X, sum joint probabilities $f(x,y)$ over all possible values of Y.

$$f_X(x) = \underbrace{P(X=x)}_{\text{Prob of just } x} = \underbrace{\sum f(x,y)}_{\text{all together}}$$

$$f_Y(y) = P(Y=y) = \sum_x f(x,y)$$

Problem 5.1

2 caplets from:

3 aspirin　Let X = aspirin
2 sedative　Let Y = sedative
4 laxitive

a) Find joint Prob?

$f(x,y) = P(X=x, Y=y)$

$S = \{ \overset{\text{none}}{(0,0)}, (0,1), (1,0), (1,1), (2,0), (0,2)$

x

y	0	1	2	
0	$\boxed{1/6}$	$1/3$	$\frac{1}{12}$	$\frac{7}{12}$
1	$\boxed{\frac{2}{9}}$	$\frac{1}{6}$	0	$7/18$
2	$\frac{1}{36}$	0	0	$1/36$

$$= \sum_x f(x,1)$$

$P(Y=1)$

$f_X(x)$:

$$f(0,0) = \frac{\binom{3}{0}\binom{2}{0}\binom{4}{2}}{\binom{9}{2}} = \boxed{\frac{1}{6}}$$

$$f(0,1) = \frac{\binom{3}{0}\binom{2}{1}\binom{4}{1}}{\binom{9}{2}} = \boxed{\frac{2}{9}}$$

In general we get:

$$f(x,y) = P(X=x, Y=y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{2-x-y}}{\binom{9}{2}}$$

Bivariate, depends on 2 variables,

$x = 0, 1, 2$
$y = 0, 1, 2$
$0 \leq x + y \leq 2$

This resembels a hypergeometric distribution.

Problem 5.1

b) Find marginal probability of X

$$\begin{array}{cccc} & x: & & \\ f_X(x): & 0 & 1 & 2 \\ & \frac{5}{12} & \frac{1}{2} & \frac{1}{12} \end{array}$$

$$\boxed{f_X(0) = \frac{5}{12}}$$

These are the coloumn totals Of the joint PMF $f(x,y)$

Marginal PMF of Y:

$$\begin{array}{cccc} & 0 & 1 & 2 \\ f_Y Y: & \frac{7}{12} & \frac{7}{18} & \frac{1}{36} \end{array}$$

Problem 5.1

c) Given $Y=1$, distribution of aspirins

$$f(x|1) = P(X=x | Y=1)$$

$$f_{X|Y}(0|1) = \frac{f(0,1)}{f_Y(1)} = \frac{2/9}{7/18} = \frac{4}{7}$$

$$f_{X|Y}(1|1) = \frac{f(1,1)}{f_Y(1)} = \frac{1/6}{7/18} = \frac{3}{7}$$

$$\underbrace{f_{X|Y}(2|1)}_{} = \frac{0}{7/18} = 0$$

makes sense, cant choose 2, given 1 is alr selected.

Practise

4 Red Let X be # of red
3 Blue Let Y be # of blue
2 Green 2 Drawn

a) Joint PMF

$$\begin{array}{cc|ccc} & & & x & \\ & & 0 & 1 & 2 \\ \hline & 0 & \frac{1}{36} & \frac{8}{36} & \frac{1}{6} \\ Y & 1 & \frac{1}{6} & \frac{12}{36} & 0 \\ & 2 & \frac{3}{36} & 0 & 0 \end{array}$$

$$= \frac{\binom{4}{x}\binom{3}{y}\binom{2}{2-x-y}}{{}^9C_2}$$

Marginal PMF for X =

$f_X(X=0) = \frac{5}{18}$

$f_X(X=1) = \frac{5}{9}$

$f_X(X=2) = \frac{1}{6}$

Marginal PMF for Y =

$f_Y(0) = \frac{5}{12}$

$f_Y(1) = \frac{1}{2}$

$f_Y(2) = \frac{1}{12}$

conditional prob

$f_{X|Y}(X|Y=1)$

$$f(0|1) = \frac{\frac{5}{18}}{\left(\frac{1}{2}\right)} = \frac{5}{9}$$

$$f(1|1) \; \frac{\frac{5}{9}}{\frac{1}{2}} = \frac{4}{9}$$

$$f_{X|Y} = (X|Y=1) = \begin{cases} \frac{5}{9} & \text{if } x=0 \\ \frac{4}{9} & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

Conditional Distribution

The conditional probability $P(X=x|Y=y)$ is given by:

$$P(X=x | Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)}$$

Divide joint PMF by Marginal PMF to get the conditional distribution.

Joint PDF Overview

The joint Pdf $f_{XY}(x,y)$ describes the probability density of 2 variables X and Y.

Joint PDF:

$$f_{XY}(x,y) \geq 0 \quad \text{for all } (x,y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Example $f_{XY}(x,y) = \begin{cases} c(x+y) & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

The PDF is only valid from 0-1. Find c s.t it integrates to 1.

$$\text{Volume} = P[(X,Y) \in A]$$

$$= \iint_A f(x,y)\,dx\,dy$$

Set up double integrals, for regions of interest

Def'n: Let X, Y be continous r.v.s with joint pdf $f(x,y)$. Then:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad \text{marginal pdf of } x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad \text{marginal pdf of } Y$$

Problem 5.2

$$P(1<X<3,\ 1<Y<2)$$

$$= \int_1^3\int_1^2 e^{-x+y}\,dy\,dx \qquad -x+2-x+1 = -2x+3$$

$$= \int_1^2 e^{-x+y}\,dy = e^{-x+y}\Big|_1^2 = e^{-x+2} - e^{-x+1}$$

$$= \int_1^3 e^{-2x+3}\,dx = e^{-2x+3}\cdot -2 = -2e^{-2x+3}\Big|_1^3 = 5.337$$

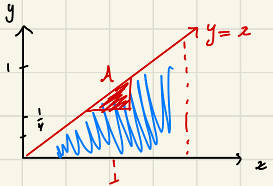
Problem 5.3

X = proportion of capacity of tank stocked at the week.

Y = proportion sold during the week

Joint d.f is $f(x,y) = \begin{cases} 3x & \text{if } 0 \leq y \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

$$P(X<\tfrac{1}{2}) \text{ and } P(Y>\tfrac{1}{4})$$



$$= \int_{\frac{1}{4}}^{\frac{1}{2}}\int_{\frac{1}{4}}^{x} 3x\,dy\,dx$$

$$= \int_{\frac{1}{4}}^{x} 3x\,dy = 3x\Big|_{y=\frac{1}{4}}^{y=x} = \int_{\frac{1}{4}}^{\frac{1}{2}} 3x - \tfrac{3}{4}\,dx = \frac{3x^2}{2} - \frac{3}{4}x\Big|_{\frac{1}{4}}^{\frac{1}{2}}$$

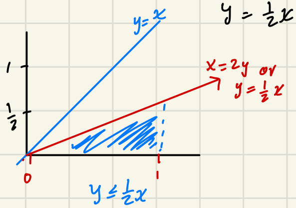
$$3x - \tfrac{3}{4}$$

$$= 0 - -\tfrac{5}{128}$$

$$= \tfrac{5}{128}$$

b) $P(X \geq 2Y)$

$$x = 2y$$
$$y = \tfrac{1}{2}x$$



$$= \int_0^{1/2}\int_{2y}^{1} 3x\,dx\,dy$$

$$= \int_0^{1/2} \frac{3x^2}{2}\Big|_{x=2y}^{1}\,dy$$

$$= \int_0^{\frac{1}{2}} \tfrac{3}{2} - 6y^2\,dy$$

$$= \tfrac{3}{2}y - 2y^3\Big|_{y=0}^{1/2} = \tfrac{1}{2}$$

Problem 5.4

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & \text{, if } 0<x<1,\ 0<y<1 \\ 0 & \text{, elsewhere} \end{cases}$$

a) $\int_0^1 \frac{2}{3}(x+2y)\,dy$

$$\tfrac{2}{3}\int_0^1 x+2y\,dy = \tfrac{2}{3}(xy+y^2)\Big|_0^1$$

$$= \tfrac{4}{3}$$

Marginal density function of x

$$f_X(x) = \begin{cases} \frac{2}{3}x + \frac{2}{3}, & \text{if } 0<x<1 \\ 0 & \text{o.w} \end{cases}$$

$$f_Y(y) = \int_0^1 \tfrac{2}{3}(x+2y)\,dx$$

$$= \tfrac{2}{3}\int_0^1 x+2y\,dx = \tfrac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_{x=0}^{x=1}$$

$$= \tfrac{2}{3}\left(\tfrac{1}{2}+2y\right) = \tfrac{1}{3} + \tfrac{4}{3}y$$

$$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{4}{3}y, & \text{if } 0<y<1 \\ 0 \end{cases}$$

Problem 5.4

b) Find $P(X \le \frac{1}{2} \mid Y \ge \frac{1}{2})$

$$= \frac{P(X \le \frac{1}{2}, Y \ge \frac{1}{2})}{P(Y \ge \frac{1}{2})}$$

$$= \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} \frac{2}{3}(x+2y)\,dx\,dy$$

$$P(X \le \tfrac{1}{2} \cup Y \ge \tfrac{1}{2}) = \frac{7}{24}$$

$$P(Y \ge \tfrac{1}{2}) = \int_{\frac{1}{2}}^{1} \underbrace{\frac{1}{3} + \frac{4}{3}y}_{f_Y(y)}\,dy$$

$$= \frac{1}{3}y + \frac{2y^2}{3}\Big|_{\frac{1}{2}}^{1} = \frac{2}{3}$$

$$P(X \le \tfrac{1}{2} \mid Y \ge \tfrac{1}{2}) = \frac{7/24}{2/3} = \frac{7}{16}$$

Def: Conditional density of X given Y = y:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$f(x,y)$: joint pdf of x & y

$f_Y(y)$: marginal pdf of Y

5.4 c)

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{2}{3}(x+2y)}{\frac{1}{3}+\frac{4}{3}y} \text{ or } \frac{2x+4y}{1+4y}$$

if $0 < x < 1$.

d) $f(x|\frac{1}{2}) = \frac{2x+2}{3}$, $0 < x < 1$

$$= \int_{0}^{1/2} f(x|\tfrac{1}{2})\,dx = \int_{0}^{1/2} \frac{2x+2}{3}\,dx$$

$$P(X \le \tfrac{1}{2} \mid Y = \tfrac{1}{2}) = \frac{5}{12}$$

Some properties

1.) $P(X \le x \mid Y = y) \ne \frac{P(X \le x, Y = y)}{P(Y = y)}$

2. $f(x|y) \ge 0$ for all x

3. $\int_{-\infty}^{\infty} f(x|y)\,dx = 1$

$$f(x,y) = \begin{cases} x+y, & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \int_{0}^{1} x+y\,dx$$

$$= \int_{0}^{1} x+y = \frac{x^2}{2} + yx\Big|_{0}^{1} = \frac{1}{2} + y$$

$f_Y(y) = \frac{1}{2} + y$ if $0 \le y \le 1$

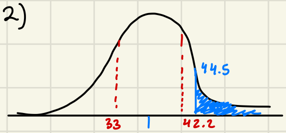
b) $$P(X \ge \tfrac{1}{2} \mid Y > \tfrac{1}{2}) = \frac{P(X \ge \frac{1}{2} \text{ and } Y > \frac{1}{2})\ P(X \ge \frac{1}{2})}{P(Y > \frac{1}{2})}$$

$$= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} x+y\,dx\,dy$$

$$= \frac{x^2}{2} + yx\Big|_{\frac{1}{2}}^{1} dy = \frac{3}{8}$$

Assignment 6- Practise

2)



$\mu = 37.6$

$\sigma = 4.6$

Normal CD is used more for height of pdf at a specific point like 36.5.

a) $P(x \geq 44.5)$ $\quad z = \frac{x-\mu}{\sigma}$

$P = 0.028156$ $\quad z = \frac{44.5-37.6}{4.6}$

$P = 0.0668$

$P(z > 1.50) = 1 - 0.9332$

$= 0.0668$

b) $P = 0.286$ $\longrightarrow$ $z = \frac{35-37.6}{4.6}$

c) $P = 0.649$

$z = -0.565$

$P = 0.28774$

Note: For these kinds of questions use Normal CD, because they are cumulative. To solve these questions first, standardize then read from table to solve for probability.

Note: For any negative Z scores, simply do $1 - P(-z)$

5) Let Z be number of aces obtained in first draw, W is total # in both draws.

a) Joint PMF:

Z can be 0, 1
W can be 0, 1, 2

W \ Z	0	1
0	$\frac{1128}{1326}$	0
1	$\frac{32}{221}$	$\frac{32}{221}$
2	0	$\frac{1}{221}$

b) $\frac{1320}{1326}$ $\quad \frac{198}{1326}$

$P(Z=0) = \frac{1320}{1326}$, $P(Z=1) = \frac{198}{1326}$

c) Conditional Given $Z=1$

$$P(W=w \mid Z=1) = \frac{P(Z=1 \cup W=w)}{P(Z=z)}$$

$$P(W=1 \mid Z=1) = \frac{\frac{192}{1326}}{\frac{198}{1326}} = \frac{96}{99}$$

$$P(W=2 \mid Z=1) = \frac{\frac{6}{1326}}{\frac{198}{1326}} = \frac{1}{33}$$

3)

3% on Z score table is $\boxed{1.89}$, 0.970

using this

$$1.89 = \frac{6-\mu}{0.05}$$

$$\mu = 6 + 0.094 = 6.0945$$

4) 23% $\quad$ Can approximate the normal using binomial,

$n = 120$

$P(x > 32)$ $\quad n = 120$

Check $np = 120 \times 0.23$

$np = 27.6 > 5$ ✓

$120(1-0.23) = 92.4$ ✓

$\mu = n \cdot p$

$\mu = 120 \cdot 0.23 \rightarrow 27.6$

$\sigma = 4.601$

Continuity for correction:

$P(X > 32) = P(X \geq 32.5) =$ $\quad z = \frac{32.5-27.6}{4.601}$

$P(Z \geq 32.5) = 0.85543$ $\quad z = 1.064$

$z = 1.$~~[illegible]~~

$1 - 0.85543$

$= 0.1446.$

6a) $$f(p,s) = \begin{cases} 5pe^{-ps}, & 0.2 < p < 0.4 \text{ and } s > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$P(p < 0.3 \text{ and } s > 2) = \int_{p=0.2}^{0.3} \int_{s=2}^{\infty} f(p,s)\, ds\, dp$$

$$P(p < 0.3 \text{ and } s > 2) = \int_{p=0.2}^{0.3} \int_{s=2}^{\infty} 5pe^{-ps}\, ds\, dp$$

$$\int_{p=0.2}^{0.30} 5e^{-2p}\, dp = \int e^{-2p}\, dp = -\frac{1}{2} e^{-2p}$$

$$= 5 \cdot \left[-\frac{1}{2} e^{-2p} \right]_{p=0.2}^{p=0.3}$$

$$\int_{p=0.20}^{0.3} 5e^{-2p}\, dp = -\frac{5}{2}\left(e^{-2(0.3)} - e^{-2(0.2)}$$

$$= -\frac{5}{2} \cdot (-0.1215) = 0.30375$$

b) $P(0.25 < p < 0.30, s < 1)$

$$= \int_{0.25}^{0.3} \int_{0}^{1} 5pe^{-ps}\, ds\, dp$$

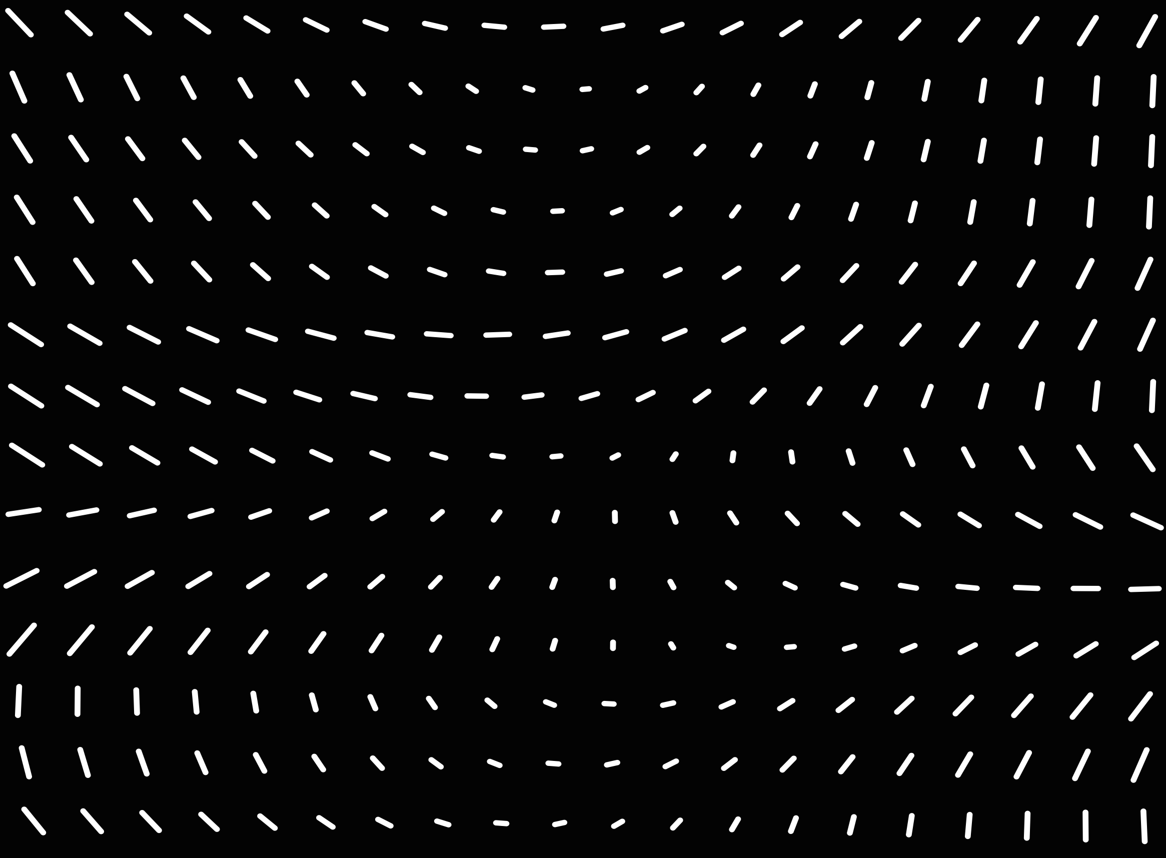
$$= \int_{0}^{1} 5p\, e^{-ps}\, ds = 5(1 - e^{-p})$$

$$= \int_{0.25}^{0.3} 5(1 - e^{-p})\, dp = 0.05$$

$$= -\left(e^{-0.30} - e^{-0.25} \right)$$

$P(0.25 < p < 0.30, s < 1) = 0.25 - 0.19 = 0.06$

Midterm 1 Study Guide



Topic 1: Probability (Sec 1.2, 2.1 - 2.8)

1.) Definitions and Terms

Probability is a measure of one's belief in the occurance of a random event.

Random Event: outcome of experiments that cannot be determined uniquely from known conditions

Classical prob. Probability of an event occuring based on equally likely outcomes.

2) Types of Rules

If A, B are mutally Exclusive

1) Addition Rule: $P(A \cup B) = P(A) + P(B) \longrightarrow$ otherwise

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Suppose we can do job 1 in n ways and job 2 in x ways. Then we can do either job 1 or job 2 in $x+n$ ways.

Also to check independent events

2.) Multiplication Rule: $P(A \cap B) = P(A) \cdot P(B)$

We can do job 1 in 20 ways
We can do job 2 in 7 ways
We can do job 1 and job 2 in 20×7 ways

3.) Permutations Rule: $P(n,r) = \dfrac{n!}{(n-r)!}$

Suppose there are n distinct objects
The # of ways to arrange (permute) r objects selected from these n objects.

(Order Matters) (objects drawn without replacement)

Combinations Rule : $C(n,r) = \frac{n!}{r!(n-r)!}$

From n distinct objects the # of ways to take out combinations of r subjects at a time is denoted by C^n_r

ORDER DOES NOT MaTTer

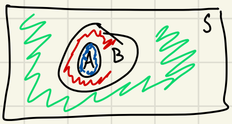
Objects are drawn without replacement

Sample Space (S) : The set of all possible outcomes of a random experiment.

Event (E): Subset of the Sample space. Event can have one or more outcomes.

Review of set notation

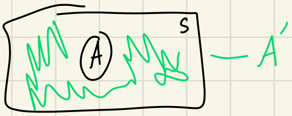
$A \subset B$ - A is a subset of B



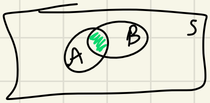
$A \cup B$ union of A and B

A or B or Both

A' or $\bar{A}$: The complement of A.



$A \cap B$



Intersection of A and B

Both A and B

$\emptyset$: empty set

De Morgan Laws:

$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$

Probability Rules

1.) $P(A) \geq 0$

2.) $P(S) = 1$

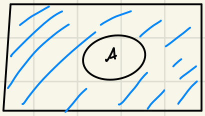
3.) $P(A \cup B) = P(A) + P(B)$ if A, B are mutually exclusive.

Thm. 2.2 An experiment can give n different outcomes, each outcome is equally likely. Event A consists of m of these outcomes.

Then $P(A) = \frac{m}{n}$

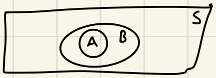
Rules of Probability

1.) $P(A') = 1 - P(A) \iff P(A') = 1 - P(A)$



2.) $P(\emptyset) = 0$ for any sample space S.

3.) A, B are events in S. and $A \subset B$.
Then $P(A) \leq P(B)$



4.) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A, B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

Conditional Probability

If A and B are two events in the sample space S, and $P(A) \not\approx 0$

The conditional probability of B given A:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Law of Probability

If A, B are two events in S, $P(A) \neq 0$.

Then: $P(A \cap B) = P(B) \cdot P(A|B)$

which is just

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ rearranged.

Independent Events

Def: Two events A, B are independent if
$(A \cap B) = P(A) \cdot P(B)$

Being independent means event B does not influence the probability of event A.

Mutually Exclusive

Def: 2 events can not occur at the same time.

$$P(A \cap B) = 0$$

Note: Mutually exclusive events can not be independent.

Because the happening of one event directly influences, the other event.

Mutual Independence

Def: A, B, C are mutually independent if:

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$
$P(A \cap B) = P(A) \cdot P(B)$
$P(A \cap C) = P(A) \cdot P(C)$
$P(B \cap C) = P(B) \cdot P(C)$

Pairwise Independence

A, B, C are Pairwise indep if:

$P(A \cap B) = P(A) \cdot P(B)$
$P(A \cap C) = P(A) \cdot P(C)$
$P(B \cap C) = P(B) \cdot P(C)$

Partition - a bunch of cases that cover all possible outcomes.

Thm 2.12 (Law of Total Probability)

If events $B_1, B_2 \ldots. B_k$ form a partition of S, and they are mutually exclusive and 1 most occur.

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \ldots\ldots P(B_k) \cdot P(A|B_k)$$

Bays Theorem

Bay's Theorem is a way of updating probabilities based on new information.

$$P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{P(A)}$$

- where $P(A|B_i)$ is the probability of A happening given B_i (likilihood)
- $P(B_i)$ is the old probability
- $P(A)$ is the total probability

Baye's Theorem is directly related to Law of total probability:

$$P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{P(A)}$$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \ldots\ldots P(B_K) \cdot P(A|B_K)$$

Longer version could be

$$P(B_i|A) = \frac{P(A|B_i) \cdot P(B_i)}{P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \ldots\ldots \text{to } B_K}$$

"With Replacement Questions"

Permutations with replacement from n^K.
Where n is # of possible outcomes and K is how many are chosen (how many positions).

Combinations with replacements $C(n+K-1, K)$
where n is # of options and K is how many your choosing.

Topic 2: Probability Distributions, Densities and Expectations

Discrete Random Variable - Can take on a countable number of distinct values

Probability Mass Function (PMF): $P(X = x)$ gives the probability that the discrete random variable X is equal to a specific value x.

Formula: $P(X = x) = p(x)$

Where $p(x)$ is the probability that the random variable X takes on the value of x. The sum of all possibilities must equal 1: $\sum P(X=x) = 1$

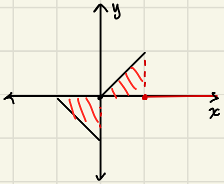
The variable X is essencially a function defined on S, that maps the outcomes of experiments to numbers.

Continous Random Variable - take values in some intervals of real #s.

Probabity Distributions for Discrete R.V's

Def: If x is a discrete r.v. Then,

$f(x) = p(X = x)$ for each x in the range of X is called the pmf of X.



Cumulative Distribution Function (CDF)

Gives the probability that a random variable X is less than or equal to a certain value.

$$F(x) = P(X \leq x)$$

This means for a value x, CDF gives the probability that X will take on a value less than or equal to x.

CDF :
$$F_X(x) = \begin{cases} 0 & \text{for } x < -3 \\ \frac{1}{8} & \text{for } -3 \leq x < -1 \\ 0.5 & \text{for } -1 \leq x < 1 \\ \frac{7}{8} & \text{for } 1 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Practise Problems:

A1.q2) 52c5 is total outcomes

a)

$$P(\text{pair}) = \frac{13c2 \cdot 4c2 \cdot 4c2 \cdot 11c1 \cdot 4c1}{52c5}$$

P(Pair)=

P(Pair) = 0.0475

b) $\frac{13c1 \cdot 4c4 \cdot 12c1}{52c5}$

P(4 Kind) = $\frac{1}{4165}$

A1.q3) {T, M, O, W, E, V}

a) 6P4 = 360

b) 4P2 = 12 → 4P2 = 12 → $12^2 = 144$

c) $\frac{\frac{4P2}{2} \cdot 4P2}{144} = \frac{1}{2}$

A1.q4) 10 possible digits
26 possible letters

a) Find all possible outcomes

$10^3 \cdot 26^3 = 17576\ 000$

P(all 3 letters) = $\frac{26 \cdot 10^3}{17576000}$

P(all 3) = 0.0015

b) P(Even or odd) = $\frac{5^3 \cdot 26^3}{1757600} + \frac{5^3 \cdot 26^3}{1757600}$

= 0.25

a1.q6) F1, F2, F3, F4, F5
C1, C2, C3

a) 5P3 = 60

b) n = 4c2 · 3!
P(A) = 0.6
or
4P2 × 3 = $\frac{36}{60}$

7) P(O) = live off campus
P(V) = live in Virginia
P(V'∪O') = live on campus or out of state

$P(O) = \frac{1}{3}$

$P(V) = \frac{5}{9}$

$P(V' \cup O') = \frac{3}{4}$

$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cap B'$

$P(V' \cap O') = ?$

$P(V' \cup O') = P(V') + P(O') - P(V' \cap O')$

$\frac{3}{4} = \frac{1}{3} + \frac{5}{9} - P(V' \cap O')$

$P(V' \cap O') = \frac{4}{9} + \frac{2}{3} - \frac{3}{4} = \frac{13}{36}$

8a) $\frac{95c4}{100c4}$

b) $\frac{5c4}{100c4}$

A2.q2) For A, B to be indep means $A \cap B = A \cdot B$

$P(A \cap B) = P(A) \cdot P(B)$

$P(A') = 1 - P(A)$

$P(B') = P(A \cap B') + P(A' \cap B')$

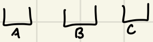
$= P(A' \cap B') + P(B') \cdot P(A)$

$P(A' \cap B') = P(B') - P(B') \cdot P(A|B') =$

$= P(B')\left[1 - P(A)\right]$

$= P(B') \cdot P(A')$

A3.q3) $\boxed{1}$ $\boxed{2}$



a) A = box 1 or box 2 empty

Total outcomes $3^2 = 9$

A = { 2 3, 13, 33, 22, 11, 31, 32 } → 12, 21

$P(A) = \frac{7}{9}$

b) $B = \frac{1}{3}$ Indep? $P(B \cap C) = P(B) \cdot P(C)$

$C = \frac{1}{3}$

$\frac{1}{9} = \frac{1}{3} \cdot \frac{1}{3}$

$\frac{1}{9} = \frac{1}{9}$

Yes B, C are indep
B, C are not mutually exclusive
because they can occur at the same time $P(A \cap B) \neq 0$

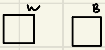
c) $D = \frac{1}{9}$ no these events are not indep since

$P(D \cap B) \neq \frac{1}{9} \cdot \frac{1}{9}$

$0 \neq \frac{1}{27}$

∴ They are mutually exclusive

A2.q4)



A = { 1st dice gives 1 or 2 or 3 }

B = { 1st dice gives 4 or 5 or 6 }

C = { sum of numbers on the dices are 9 }

a) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

This is proving mutual independence.

6^2 options = 36 ways: $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$

to make 9 is {6 3, 36, 45, 54} $\frac{4}{36}$ $P(C) = \frac{1}{9}$

The one way to do A, B, C is 36 ∴ $P(A \cap B \cap C) = \frac{1}{36}$

$\frac{1}{36} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{9}$

$= \frac{1}{36}$

More Practise

A2.q4)

b) To Check independence of 3 events, they must be Pairwise independent. They dont rely on each other for outcomes

$P(A \cap B) = \frac{1}{6}$

$P(A \cap B) = P(A) \cdot P(B)$

$\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{2}$

$\frac{1}{6} \neq \frac{1}{4}$ $\therefore$ They are not independent.

a2.q6

6) $100C2$ - Total outcomes

a) $\frac{25C1 \cdot 40C1}{100C2}$

$P(R+W) = \frac{20}{99}$

b) $\frac{25}{100} \cdot \frac{40}{100} + \frac{40}{100} \cdot \frac{25}{100} = 0.2$

a2q5)

$\frac{1}{50}$ See a given magazine add

$\frac{1}{5}$ Sees the corresponding magazine too

$\frac{1}{100}$ Sees both

$\frac{1}{3}$ Purchases after seeing add

$\frac{1}{10}$ Purchase without Seeing add

Let S be PPl who see the add

Let B be PPl who bought

Let A be PPl who saw TV. add

Let M be PPl who got magazine add

Given:

$P(B|S) = \frac{1}{3}$

$P(B|S') = \frac{1}{10}$

$P(M) = \frac{1}{50}$

$P(T) = \frac{1}{5}$

$P(M \cap T) = \frac{1}{100}$

$P(B) = ?$

$P(B|S) = \frac{P(B \cap S)}{P(S)}$

$P(B \cap S) = P(B|S) \cdot P(S)$

$P(S') = 1 - \frac{21}{100}$

Then $P(S) = P(M \cup T)$

$P(M \cup T) = P(M) + P(T) - P(M \cap T)$

$P(M \cup T) = \frac{1}{50} + \frac{1}{5} - \frac{1}{100}$

$P(M \cup T) = \frac{21}{100}$

$P(S) = \frac{21}{100}$

$P(B) = P(B \cap S) + P(B \cap S')$

$P(B) = P(B|S) \cdot P(S) + P(B|S') \cdot P(S')$

$P(B) = \frac{1}{3} \cdot \frac{21}{100} + \frac{1}{10} \cdot \frac{79}{100}$

$P(B) = \frac{149}{1000}$

$P(B) = 0.149$

7) Let D represent a person over 50 having diabetes
Let C represent a Correct diagnosis or a pass of diabetes test

$P(D) = 0.08$

$P(C|D) = 0.95$

$P(C|D') = 0.02$

a) $P(C) = ?$

$P(D|C) = \frac{P(C \cap D)}{}$

$P(C|D) = \frac{P(C \cap D)}{P(D)}$

$P(C \cap D) = 0.08 \cdot 0.95$

$P(C \cap D) = 0.076$

$P(C) = P(C|D) \cdot P(D) + P(C|D') \cdot P(D')$

$P(C) = 0.95 \cdot 0.08 + 0.02 \cdot 0.92$

$P(C) = 0.0944$

b) $P(D|C) = \frac{P(C|D) \cdot P(D)}{P(C)}$

$P(D|C) = \frac{0.95 \cdot 0.08}{0.0944}$

$P(D|C) = 0.805$

8)

0.05	0.1	0.25	0.6
0	1	2	3

$P(B|A_0) = 0$

$P(B|A_1) = \frac{1}{3}$

$P(B|A_2) = \frac{2}{3}$

$P(B|A_3) = 1$

$P(A_3|B) = \frac{P(B|A_3) \cdot P(A_3)}{P(B)}$

$= \frac{1 \times 0.6}{0.8} = 0.75$

where $P(B) = P(B|A_1)P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3) \cdot P(A_3)$

$P(B) = \frac{1}{3} \times 0.1 + \frac{2}{3} \times 0.25 + 1 \times 0.6 = 0.8$

$P(A_3|B) = 0.75$

Practise STAT 268 Midterm 1.

1a) $\{1, 2, 3, 4, 5\}$

$P(E \cap F) = ?$

$P(E) = 8/20 = 0.4$

$P(E) = \{12 \quad 14 \quad 24 \quad 32 \quad 34 \quad 42 \quad 52 \quad 54\}$

$P(F) = \{31, 32, 34, 35, 41, 42, 43, 45, 51, 52, 53, 54\}$

$= \frac{12}{20} = 0.6$

Are P(E) and P(F) independent?

No, $P(E \cap F) \neq P(E) \cdot P(F)$

$P(E \cap F) = \frac{1}{4} = 0.25$

b) Dice 1: $\{1, 2, 3, 4, 5, 6\}$ *

Dice 2: $\{1, 2, 3, 4, 5, 6\}$ *

Sample Space $S: \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

36 ways

$P(E \cap F) = \frac{1}{3}$

$P(E) = 0.5$

$P(F) = \frac{2}{3}$

If $P(E \cap F) = P(E) \cdot P(F)$ They are independent.

$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$ ✓

∴ Indep

3) $52C5 = 2598960$

a) 3 of a Kind $= 12C1 \cdot 4C3$
$= 48$

2 Aces: $\cdot\, 4C2$

$$= \frac{12 \times 4C3 \times 4C2}{52C5}$$

b) $52C13$

4 aces : $4 \times 4C4 \cdot 48C9$

$\overset{\text{amt of ppl who can get 4 Aces}}{4}$

$48C9 \longrightarrow$ ways to select remaining cards

$$= \frac{4 \times 48C9}{52C13}$$

$$= \frac{44}{4165}$$

P(B)= Means somebody is left handed

4) $P(A_0)$ means he did the crime

$P(A_0) = 0.6 \quad P(B|A) = 1$

$P(A|B) = ?$

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|A') \cdot P(A')}$$

$P(B|A) \cdot P(A) = 1 \cdot 0.6$

$P(B|A') \cdot P(A') = 0.2 \times 0.4$

$$P(A|B) = \frac{1 \cdot 0.6}{1 \cdot 0.6 + 0.2 \times 0.4}$$

$P(A|B) = 0.882$

The new Probability is 88.2%.

2) $\{A, B, C, D\}$

A, B - 2 hats

$4C2 = 6$

$S = \{AB, AC, AD, BC, BD, CD\}$

a) P(connor or David) $= \frac{4}{6} = \frac{2}{3}$

b) Order matters: $4P2 = 12$

AB

$\{AB, BA, AC, CA, AD, DA, BC, BD, CB, CD, DB, DC\}$

$f(x) =$	x	0	1	2
$P(x=0) =$		$\frac{7}{12}$	$\frac{4}{12}$	$\frac{1}{12}$

Practise Stat 268 Midterm

1a) $P(A)=0.25$ $P(A')=0.75$

$P(S)=0.35$ $P(S')=0.65$

$P(A\cap S')=0.1$

$P(A'\cup S')=?$

$P(A) = P(A\cap S') + P(A\cap S)$

$0.25-0.1 = P(A\cap S)$

$0.15 = P(A\cap S)$

$P(A\cap S) = P(A) + P(S) - P(A\cup S)$

$P(A\cup S) = 0.25 + 0.35 - 0.15$

$P(A\cup S) = 0.45$

$P(A'\cup S') = 1 - P(A\cup S)$

$P(A'\cup S') = 1 - 0.45$

$= 0.55$

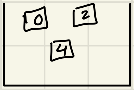
b) $P(A\cap S) = P(A)\cdot P(S)$

$P(A\cap S) = 0.25\cdot 0.35$

$0.15 \neq 0.0875$

$\therefore$ not indep

2)



$0=0$
$2=200$
$4=400$

A — Chooses $_3C_1$

a) 3^2 ways to choose

$S = \{A_0B_0, A_2B_2, \underline{A_4B_4}, A_0B_2, \underline{A_0B_4}, \underline{A_2B_4}, \underline{A_4B_0}, \underline{A_4B_2}, A_2B_0\}$

$P(E) = \frac{5}{9}$

b) 5^2 ways

$P(\text{Not getting } 4): \frac{4}{5}\times\frac{4}{5} = \frac{16}{25}$

$P(4) = 1 - \frac{16}{25}$

$P(4) = \frac{9}{25}$

3) $P(D)=0.001$ $P(D')=0.999$

a) $P(P\mid D) = 0.98$

$P(P\mid D') = 0.01$

$P(P) = ?$

$$P(P\mid D) = \frac{P(P\cap D)}{P(D)}$$

$P(P\cap D) = P(D)\cdot P(P\mid D)$

$P(P\cap D) = 0.001 \cdot 0.98$

$P(P\cap D) = 0.00098$

$P(P) = 1 - (P(P\mid D) + P(P'\mid D'))$

$P(P) = P((P\mid D)\cdot P(D) + P(P\mid D')\cdot P(D')$

$P(P) = 0.98\times 0.001 + 0.01\cdot 0.999$

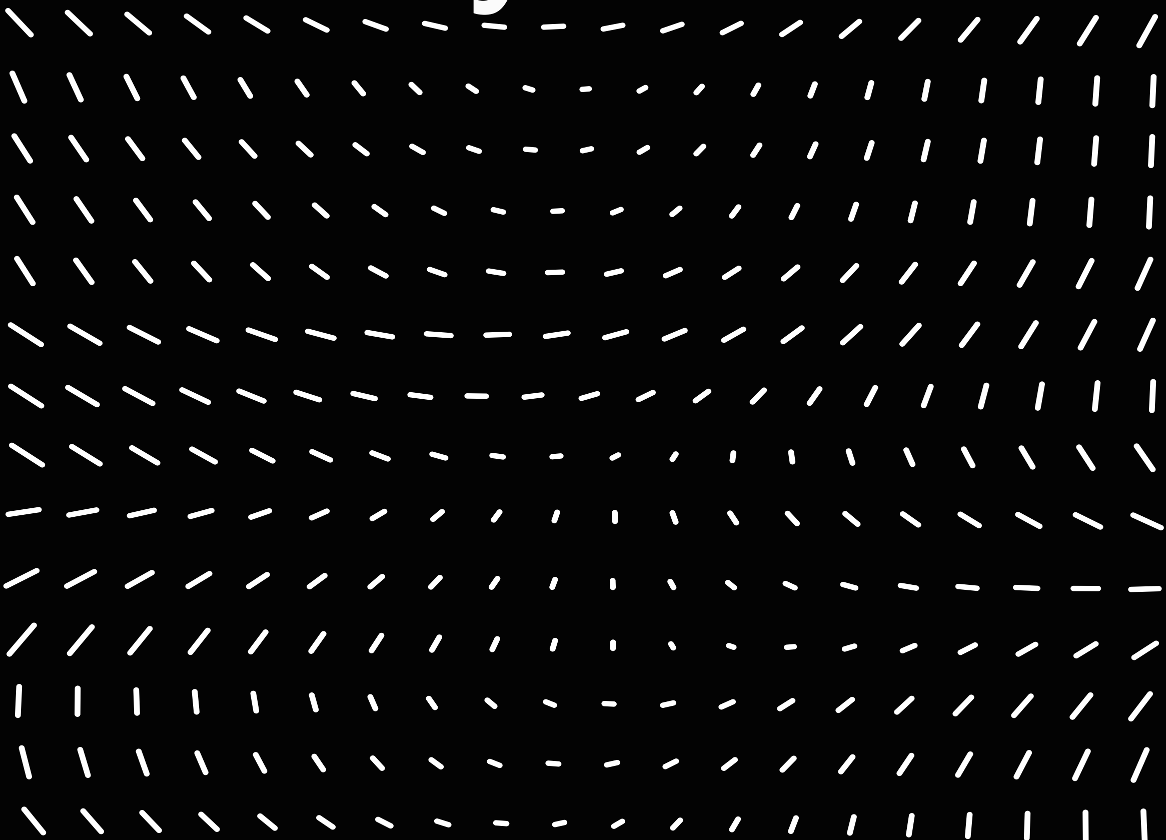
$P(P) = 0.01097$

3b) $$P(D\mid P) = \frac{P(P\mid D)\cdot P(D)}{P(P)}$$ Bayes Theorem

$$P(D\mid P) = \frac{0.98\cdot 0.001}{0.01097}$$

$P(D\mid P) = 0.0089334 5$

Midterm 2 Study Guide



Topic 2: Probability Distributions, Densities and Math expressions

Random Variable – function that assigns a numerical value to each outcome

Discrete R.V – Takes on countable number of possible values

Continous R.V – Takes on an infinite number of possible values in a range.

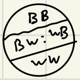
Probability Distribution's for Discrete R.V's

Def: If x is a discrete r.v, then $f(x) = P(X = x)$ for each x in the range of X is called the pmf **Probability mass function** of x

Pr-2.1

- 5 black
- 3 white

a) S



$x = \{\#\text{ of black}\}$

$x = 0, 1, 2$

$$f(0) = P\{ww\} = \frac{\binom{3}{2}}{\binom{8}{2}} = \frac{3}{28}$$

$$f(1) = P(x=1) = P\{Bw, wB\} = \frac{\binom{5}{1}\cdot\binom{3}{1}}{\binom{8}{2}} = \frac{15}{28}$$

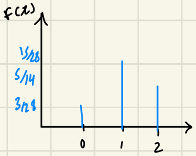
$$f(2) = P(x=2) = P\{BB\} = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{5}{14}$$

To confirm all possible outcomes $= 1$

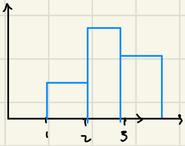
Prof of x:

x	0	1	2
$f(x)$	3/28	15/28	5/14

Graph your PmF



or



1 b) $P(\text{matching pair}) = P(x=0 \text{ or } x=2)$

$$= \frac{3}{28} + \frac{5}{14} = \frac{13}{28}$$

c) $F(x) = \frac{\binom{5}{x}\binom{3}{2-x}}{\binom{8}{2}}$; $x = 0, 1, 2$

$P(x=x)$

(Hyper geometric distribution)

Thm 3.1. For any discrete R.V x. Let $F(x)$ be its P inF. Then

1. $0 < F(x) \leq x$, for any possible x value
2. $\sum_{\text{all } x} f(x) = 1$

Cumulative Distribution Function (CDF)

Def: If x is a discrete r.v. It's CDF is given by: $F(x) = P(X \leq x)$

Gives the probability x will take a value **less than or eaual** to.

EX

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{3} & \text{for } x \leq 3 \\ \frac{2}{3} & \text{for } x \leq 5 \\ 1 & \text{if } x \geq 6 \end{cases}$$

The Probability mass function

Is the function that maps out the possibilities

EX

x:	0	1	2	(Adds to 1)
$f(x)$ $(P(x=x))$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$	

Frequency Distribution for Observed Data

Observed Data - data one has seen

Frequency - how many times that number has occurred

Types of Frequency

Absolute Freq - Actual number of times a value occurs

Relative Freq - $\dfrac{\text{Absolute Freq}}{\text{Total \# of Data pts}}$

Pr. 2.5 b)

$2^4 = 16$ Possible outcomes

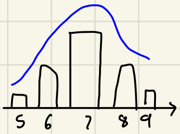
PmF

x:	0	1	2	3	4
$f(x)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Binomial Distribution

Pr 2.6 a)

5 | 9
6 | 4 7
7 | 1 4 6 8
8 | 2 5
9 | 2

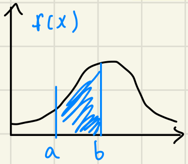


Norma Distribution of frequencies

Continous R.V, PDF and CDF.

Def. For a continous random Variable X, the PDF $f(x)$ gives the Probability that X falls within a certain range. The probability that x lies between a and b is:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$



area under the graph.

Note: ① If x is cont r.v then $P(X=x)=0$

② The total area under the pdf curve is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

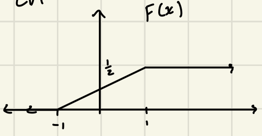
Problem 2.9

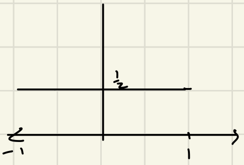
$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

If $x < -1$, C

If $x \ge 1$, x

CDF





$$f(x) = \frac{d\,F(x)}{dx}$$

To get P.D.F, we differentiate CDF

Note: $\int_0^1 5 \rightarrow 5x + C$

PDF → CDF
Integrate

CDF → PDF
Derivative

Problem 2.10

$$f(x) = \begin{matrix} -x & -1 < x < 0 \\ x & 0 \le x < 1 \\ 0 & \text{otherwise} \end{matrix}$$

Integrate to find CDF

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ & \text{if } -1 < x < 0 \end{cases}$$

For $-1 < x < 0$

$$\int_{-1}^{x} -t\,dt$$

$$= -\frac{t^2}{2}\Big|_{1}^{x} = \frac{1}{2} - \frac{x^2}{2}$$

For $0 < x$ 1

$$\int_{-1}^{0} x \,dx + \int_{0}^{\infty} x\,dx$$

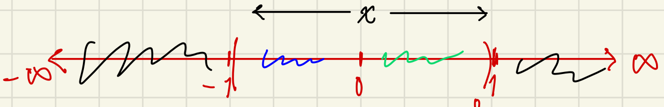
$$\frac{1}{2} + \left(\frac{t^2}{2}\Big|_0^x\right)$$

$$= \frac{1}{2} + \frac{x^2}{2}$$

$$F(x) = \begin{cases} 0, \text{ if} & x < -1 \\ \frac{1}{2} - \frac{x^2}{2} & \text{if } -1 \le x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Note: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

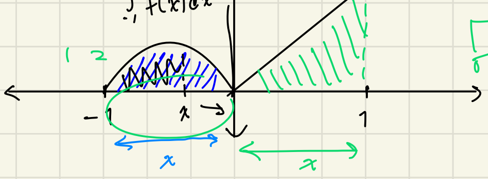
Integrate up to x, not bounds.

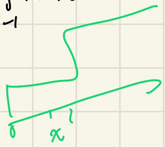


$$\int_{-1}^{x} f(x)\,dx \overset{??}{=} \int_{-1}^{0} f(x)\,dx$$

$$\lim_{x \to 0} \int_{-1}^{x} f(x)\,dx \; , \; \int_{-1}^{0} f(x)\,dx = C$$

$$\int_{-1}^{x} f(x)\,dx$$





$$\int_{-1}^{x} f(x)\,dx = C$$

$\int$

Mathematical Expectations

Expected Value: The long run average value of a R.V, if you were to do an experiment many times.

For Discrete RV: Sum all possible outcomes, weighted by probability

$$E(X) = \sum_{x} x \cdot P(X=x)$$

For continous RV: Integrate the values, weighted by densities

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Problem 2.11

a)

x:	1	-1
$f(x)$:	$1-\left(\frac{5}{6}\right)^4$ 0.518	$\left(\frac{5}{6}\right)^4$

$$P(x=-1) = \left(\frac{5}{6}\right)^4$$

$$E(x) = 1 \cdot 0.518 + (-1)\left(\frac{5}{6}\right)^4$$

$$= 0.036 \qquad > 0$$

∴ So he is expected to win a bit.

Properties of Expectations

1. $E(b) = b$, The expectation of a constant is simply the constant

2. $E(aX+bY) = aE(X) + bE(Y)$

If X and Y are r.v and a, b are constants, you can split them up.

3. $E(X+Y) = E(x) + E(Y)$

The expected value of a sum is sum of expected values

Problem 2.13

J, Q = \$15
K, A = \$5
2,3,4,5,6,7,8,9,10 = −\$4

X = money he makes from 1 draw

x:	15	5	-4
$f(x)$:	$\frac{8}{52}$	$\frac{8}{52}$	$\frac{36}{52}$

$$E(X) = 15 \cdot \frac{8}{52} + 5 \cdot \frac{8}{52} + (-4)\frac{36}{52}$$

$$E(x) = ¢0.31$$

Note: This is a discrete random variable, this means he is expected to win 31¢ on avg.

Problem 2.14

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)}\, dx \xrightarrow{\quad} \text{how to integrate}$$

$$U = 1+x^2$$

$$dU = 2x$$

$$\int \frac{2\,dU}{\pi \cdot U}$$

$$= \frac{2}{\pi} \ln U$$

$$= \frac{2}{\pi} \ln(1+x^2) \Big|_{x=0}^{1}$$

$$= \frac{\ln 4}{\pi}$$

$$= 0.4413$$

Theorem 4.1

- Expectation of a function of a r.V

If x is a discrete r.V with pmf $f(x)$,

$$E[g(x)] = \sum_{\text{all } x} g(x)\cdot f(x)$$

So, we apply some function g to X before taking Expectation

- If X is a con't r.v with pdf $f(x)$,

then $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f(x)\cdot dx$

Problem 2.15

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $E[X] = \int_0^1 x\cdot 2x\, dx = \frac{2x^3}{3}\Big|_0^1$

$= \frac{2}{3}$

b) $Y = 200X^{\frac{1}{2}} + 60$

$E[Y(X)] = \int_0^1 (200X^{\frac{1}{2}} + 60)\cdot 2x\,dx$

$= \$220$

c) $P(Y > 180) = P(200x^{\frac{1}{2}} + 60 > 180)$

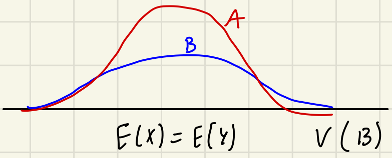
$= P(200x^{\frac{1}{2}} > 120)$

$= P\left(x^{\frac{1}{2}} > \frac{120}{200}\right)$

$= P(x > (0.6)^2)$

$= \int_{0.36}^{1} 2x\,dx = x^2\Big|_{0.36}^{1}$

$= 0.8724$



Moments of a Random Variable

- a moment is a way to describe various aspects of a distribution of r.v X.
- The r'th moment of X is $\mu'_r = E(X^r)$
- Moments are used to understand shape and characteristics of distributions.

Ex

First Moment $(r=1)$: $E(X)$, the mean of X.

Second Moment $(r=2)$. $E(X^2)$

Central Moments:

The r-th central moment is $\mu_r = E[(X-\mu)^r]$

also called r'th moment about the mean.

Variance and Standard Deviation

The Variance is a measure of how much X deviates from it's mean μ.

This is the averaged square distance of X from it's mean.

- $Var(X) = E[(X-\mu)^2]$

$V(X) = \sigma^2$

$\sqrt{V(X)}$: is called standard deviation of X. Denoted by σ.

Theorem 4.6

$$V(X) = E(x^2) - \boxed{[E(x)]^2}$$

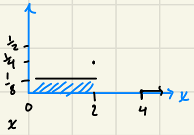
μ is a constant

$$= E[(X-\mu)^2]$$
$$= E[x^2 - 2\mu x + \mu^2]$$
$$= E(x^2) - E[2\mu x] + E(\mu)^2$$
$$= E(x^2) - 2\mu E(X) + \mu^2$$
$$= E(x^2) - 2\mu^2 + \mu^2$$
$$= E(x^2) - \mu^2$$

Problem 2.17

a) $$f(x) = \begin{cases} \frac{1}{8} & \text{if } 0 \le x < 2 \\ \frac{x}{8} & \text{if } 2 \le x < 4 \\ 0 & \text{otherwise} \end{cases}$$

Note: $f(x) = \frac{2}{8} = \frac{1}{4}$ and @ $x=4$ $= \frac{1}{2}$



CDF $F(x)$

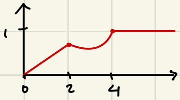
To find CDF, we integrate

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \frac{1}{8} dx = \frac{x}{8} & \text{if } 0 \le x < 2 \\ \int_0^2 \frac{1}{8} dx + \int_2^x \frac{t}{8} dt = \frac{1}{4} + \frac{x^2}{16}\Big|_2^x = \frac{x^2}{16} & \text{if } 2 \le x < 4 \\ 1 & \text{if } x \ge 4 \end{cases}$$

Is it continous?

Yes, it is.

$x=2$, $F(x) = \frac{1}{4}$



To be continous:

- Non negative
- Adds up to 1

Variance: $V(X) = E[(X-\mu)^2]$

Standard Deviation: $= \sigma = \sqrt{V(X)}$

$$V(X) = E(x^2) - [E(x)]^2$$

Problem 2.16

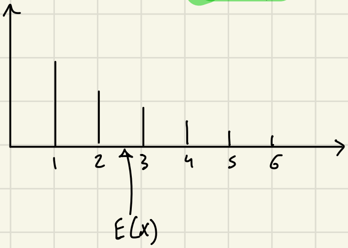
Smaller #: 1

X = min of the Pair of #'s

Pmf

x:	1	2	3	4	5	6
$f(x)$:	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + \dots + 6 \cdot \frac{1}{36}$$
$$= 2.54$$



$$V(X) = E(x^2) - [E(X)]^2$$

$$E(X^2) = \sum_{\text{all } x} x^2 \cdot f(x) = 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + \dots 6^2 \cdot \frac{1}{36}$$
$$= 8.36$$

$$V(X) = 8.36 - (2.54)^2 = 1.91$$

Problem 2.17b)

Expectations and Variance

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$
$$= \int_0^2 x \cdot \frac{1}{8} dx + \int_2^4 x \cdot \frac{x}{8} dx$$
$$= \frac{x^2}{8 \cdot 2}\Big|_0^2 + \frac{x^3}{3 \cdot 8}\Big|_2^4 =$$
$$\frac{4}{16} + \left[\frac{8}{3} - \frac{1}{3}\right]$$

$$E(x) = \frac{31}{12}$$

$$V(x) = E(x^2) - [E(x)]^2$$

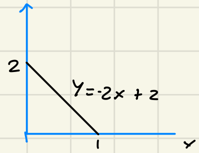
$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$
$$= \int_0^2 x^2 \cdot \frac{1}{8} dx + \int_2^4 x^2 \cdot \frac{x}{8} dx$$
$$= \frac{47}{6}$$

$$V(x) = \frac{47}{6} - \left(\frac{31}{12}\right)^2$$

$$V(x) = 1.16$$

Problem 2.18

$$f(x) = \begin{cases} 2(1-x) & , \text{ if } 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$



a) Show $\mu'_r = \frac{2}{(r+1)(r+2)}$

$$E[x^r] = \int_{-\infty}^{\infty} x^r f(x)dx = \int_0^1 x^r \cdot 2(1-x)dx$$

$$= \int_0^1 2x^r(1-x)dx = 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2} \right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2} \right] = \frac{2}{(r+1)(r+2)}$$

b) Using the Eq. $= \frac{2}{(r+1)(r+2)}$

$$V(x) = E(x^2) - [E(x)]^2$$

$$V(x) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

Notes:

1.) $V(X) \geq 0$

2.) $E(X^2) \geq [E(x)]^2$

3.) $V(aX+b) = a^2 \cdot V(X)$ for any constant a, b

Problem 2.19

$$f(x) = \begin{cases} \frac{1}{2}e^{-\frac{(x-3)}{2}} & , \text{ if } x > 3 \\ 0 & \text{otherwise} \end{cases}$$

a) $E(x) = \int_{-\infty}^{\infty} x \cdot f(x)dx$

Let $u = x-3$
$x = u+3$
$dx = du$

$$E(x) = \int_3^{x} x \cdot \frac{1}{2}e^{\frac{-(x-3)}{2}}$$

$$E(x) = \frac{3}{2}\int_0^{\infty} ue^{-\frac{3}{2}}du + \frac{3}{2}\int_0^{\infty} 3e^{-\frac{3}{2}u}du$$

$$E(x) = \frac{3}{2}\left(\frac{8}{9} + 3 \cdot \frac{2}{3}\right) = 5$$

$$E(x^2) = \int_3^{\infty} x^2 f(x)dx = \int_3^{\infty} x^2 \cdot \frac{3}{2}e^{-\frac{3}{2}(x-3)}dx$$

$$E(x^2) = 29$$

$$\text{var}(x) = 29 - 5^2 = 4$$

b) $P(1 < X < 9) = 1 - e^{-3} = 0.95$

Chebyshev's Theorem

Provides a way to estimate the probability that a random variable lies within a certain number of standard deviations from the mean.

For any random variable X with mean μ and variance σ^2, for any $k > 1$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

or

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

k is the number of std's away it is from mean.

Note: This theorem provides a lower bound for the prob that X takes values within k std of the mean

Problem 2.20

a) $E(X) = 5$

$V(X) = 4 \rightarrow \sigma^2 = 4 \rightarrow \sigma = 2$

$P(1 < X < 9) = ? \longrightarrow P(-4 \leq X - 5 \leq 4)$

$P(|X - \mu| < K\sigma) \geq 1 - \frac{1}{K^2}$

$P(1 < X < 9) = P(|X-5| < 4)$ Equal

$P(|X-5| < 4) = 1 - P(|X-5| \geq 4)$

Since $\sigma = 2$, we set $K\sigma = 4 \therefore K = 2$

$P(|X-5| \geq 4) \leq \frac{1}{K^2} = \frac{1}{4}$

$P(|X-5| < 4) = 1 - \frac{1}{4} = 0.75$

b) $P(X \leq 2 \text{ or } X \geq 8)$

$= P(X-5 \leq -3 \text{ or } X-5 \geq 3)$

$= P(|X-5| \geq 3)$

Set $K\sigma = 3$, $K = \frac{3}{2} = 1.5$

$P(|X-5| \geq 3) \leq \frac{1}{K^2} = \frac{1}{(1.5)^2} = \frac{1}{2.25}$

$P(X \leq 2 \text{ or } X \geq 8) = 0.44$

$K = \frac{L - \mu}{\sigma}$

$K = \frac{9-5}{2}$

$K = 2$

$1 - \frac{1}{K^2}$

$1 - \frac{1}{4} = 0.75$

$\frac{8-5}{2} = \frac{3}{2}$

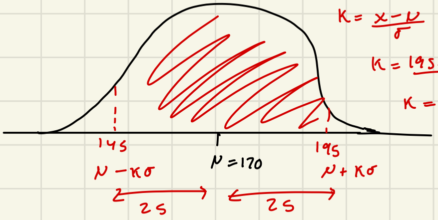
$K = \frac{3}{2}$

$1 - \frac{1}{(1.5)^2}$

$= 0.44$

Notes on Chebyshev's Thm

- Applicable on any type of distribution
- Giveaway to use it, is the two bounds given are equal distant apart



$K = \frac{x - \mu}{\sigma}$

$K = \frac{195 - 170}{10}$

$K = 2.5$

$1 - \frac{1}{K^2} = 1 - \frac{1}{2.5^2} = 0.84$

Moment Generating Function

Recall that $N_r' = E(x^r)$, $r=1,2,3$

If two random variables X and Y have the same r^{th} moments for all $r=1,2,3$ then X and Y have the same probability distribution.

For a random variable X with pdf $f(x)$ the mgf is defined as $M_x(t)=E[e^{tx}]$

For Continous R.V:

$$M_x(T) = \int_{-\infty}^{\infty} e^{tx}\cdot f(x)\,dx$$

The moment generating function allows us to find all the different moments ($E[x^2], E[x^{53}]$ etc)

Finding the first derivative of the MGF and setting it to 0 gives you First Moment

Second Derivative Set to 0, gives the Second moment.

First Moment: $\left.\frac{d\,m(t)}{dt}\right|_{t=0} = E(x)$

second moment $\left.\frac{d^2 m(t)}{dt^2}\right|_{t=0} = E(x^2)$

In general, $\left.\frac{d^r \cdot m(t)}{dt^r}\right|_{t=0} = E(x^r) = N_r'$

---

problem 2.21

$f(x) = \frac{\binom{5}{x}\binom{3}{2-x}}{\binom{8}{2}}$ for $x=0,1,2$

1) Find pmf, by plugging in x values
2) Multiply e^{tx} times the probability of x's.

a) $M_x(t) = E(e^{tx}) = \sum e^{tx}\cdot f(x)$

Pmf: x :	0	1	2
$f(x)$:	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

$= e^{t\cdot 0}\cdot f(0) + e^{t\cdot 1}\cdot f(1) + e^{t\cdot 2}\cdot f(2)$

$= \frac{3}{28} + \frac{15}{28}\cdot e^t + \frac{5}{14}e^{2t}$

b) Finding $E(X)$ and $V(X)$ through the mgf.

set $d\,m(t) = 0$

$E(x) = \left.\frac{dm(t)}{dt}\right|_{t=0} = \frac{15}{28}e^t + \frac{5}{14}\cdot 2e^{2t}$

$= \frac{15}{28} + \frac{10}{14} = \frac{35}{28} = \frac{5}{4}$

$E(x) = \frac{5}{4}$

Second derivative:

$= \left.\frac{15}{28}e^t + 4e^{2t}\cdot\frac{5}{14}\right|_{t=0}$

$= \frac{15}{28} + \frac{10}{7}$

$E(x^2) = \frac{55}{28}$

$V(x) = E(x^2) - [E(x)]^2$

$V(x) = \frac{55}{28} - \left(\frac{5}{4}\right)^2$

$V(x) = \frac{45}{112}$

Problem 2.22

$$f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) For a continous r.V

$$m_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{1}{2} e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{tx} \cdot e^{-\frac{x}{2}}\,dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{\left(t - \frac{1}{2}\right)x}$$

$$= \frac{1}{2} \left. \frac{e^{\left(t-\frac{1}{2}\right)x}}{2t-1} \right|_{x=0}^{\infty}$$

The mgf must be a finite value

$$= \lim_{x \to \infty} \frac{e^{\left(t-\frac{1}{2}\right)x}}{2t-1} - \frac{1}{2t-1}$$

$$m(t) = \begin{cases} \frac{1}{2t-1} - \frac{1}{2t-1} = 0 & \text{if } t = \frac{1}{2} \\ -\frac{1}{2t-1} = \frac{1}{1-2t} & \text{if } t < \frac{1}{2} \\ \infty & \text{if } t > \frac{1}{2} \end{cases}$$

The m.g.f of X is $m(t) = \frac{1}{1-2t}$ for $t < \frac{1}{2}$.

b) $\frac{d\,m(t)}{dt} = \left.\frac{2}{(1-2t)^2}\right|_0$ $\quad \frac{d m^2(t)}{dt^2} = \frac{6}{(1-2t)^3}$

$E(x) = 2$

$E(x^2) = 8$

$V(X) = E(x^2) - \left[E(x)^2\right]$

$V(x) = 8 - 2^2$

$V(x) = 4$

If two r.vs have the same m.g.f they must have the same distribution.

Problem 2.23

$$m(t) = \frac{1}{6}\boxed{\,} + \frac{2}{6} e^{\boxed{2}t} + \frac{3}{6} e^{\boxed{3}t}$$

$$m_x(t) = E\left[e^{tx}\right] \longrightarrow \text{By Definition}$$

The random variables take on the values of 0, 2, 3 because these are the coeffecient on the t terms.

The terms suggest : $P(X=0) = \frac{1}{6}$

$P(X=2) = \frac{2}{6}$

$P(X=3) = \frac{3}{6}$

The Pmf gives You probabilities directly
The mgf reperesents these probabilities in a transformed way.

PMF is directy related to the MGF.

Topic 3

• Bernoulli Distribution and Binomial Distribution

Definition: A bernoulli trial is an experiment with 2 mutually exclusive distinct outcomes. There are Successes (P) and Failures (1 − P).

Example: Toss a coin, Shoot target...etc

Let x = # of success in a bernoulli trial

$$x = \begin{cases} 1, & \text{if success} \\ 0 & \text{failure} \end{cases}$$

$P(x=1)=0$ $\quad$ $p(x=0)=1-0$

Probability Mass Function For Bernoulli random variable X

$$P(X=x) = p^{x}(1-p)^{1-x}, \text{ where } x=0,1$$

Way to write: $X \sim \text{Bernoulli}(\theta)$

Binomial Distribution

- A binomial experiment consists of n identical and independent Bernoulli trials

Binomial Random Variable: $x \sim \text{Binomial}(n, p)$

PMF of Binomial Distribution

For a binomial random variable X, the probability of getting exactly K successes in n trials is $P(X=K) = \binom{n}{K} p^{K}(1-P)^{n-K}$.

Binomial Distribution

1. Consists of n identical trials
2. independent trial
3. Each trial is a Bernoulli trial
4. Prob of success is θ in each trial

Problem 3.1

$\frac{2}{5}$ have O+

$\frac{1}{15}$ have O−

3 randomly selected

X # of donors with O+ blood

Recognize, $x \sim Bin(n=3, \theta = \frac{2}{5})$

n is # of trials, θ is probability of success.

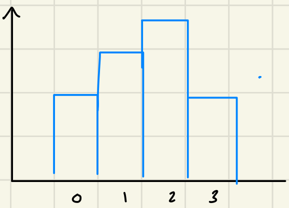
Values that x can take on, X: 0, 1, 2, 3.

Find PMF: $f(0) = P(X=0) = (1-\frac{2}{5})^3$

$P(X=1) = \binom{3}{1}\cdot(\frac{2}{5})(1-\frac{2}{5})^2$

$P(X=2) = \binom{3}{2}(\frac{2}{5})(1-\frac{2}{5})$

$P(X=3) = (\frac{2}{5})^3$



Using the Pmf of binomial dist. $P(X=k) = \binom{n}{k} p^k (1-P)^{n-k}$. $x = 0, 1, 2, 3$

$f(x) = \binom{3}{x}(\frac{2}{5})^x(1-\frac{2}{5})^{3-x}$,

Note: Bernoulli(θ) is a special case of Bin(n, θ) with n=1.

Thm 5.4

The mgf of $x \sim Bin$ is $m(t) = [1+\theta(e^t-1)]^n$

Proof:

$$M(t) = E(e^{tx})$$
$$= \sum_{x=0}^{n} e^{tx}\binom{n}{x}\theta^x(1-\theta)^{n-x}$$
$$= \sum_{x=0}^{n}\binom{n}{x}(\theta e^t)^x(1-\theta)^{n-x}$$
$$= [1+\theta(e^t-1)]^n$$

Thm

If $x \sim Bin(n, \theta)$, then $E(x) = n\theta$, $V(x) = n\theta(1-\theta)$

Pf: $\frac{dm(t)}{dt}\Big|_{t=0} = n\theta$

$$\frac{d^2m(t)}{dt^2} = n(n-1)[1+\theta(e^t-1)]^{n-2}\theta e^t \cdot \theta e^t + n[1+\theta(e^t-1)]^{n-1}\theta e^t$$

$$E(x^2) = n(n-1)\theta^2 + n\theta$$

$$V(x) = n(\theta-\theta^2) = n\theta(1-\theta)$$

Problem 3.2

Recovery rate 0.3

15 People selected (n trials)

a) Let X be the # of People among the 15 who recovered.

$x \sim Bin(n=15, \theta=0.3)$

a) $P(x \leq 3) = P(X=0) + P(x=1) + P(X=2) + P(X=3)$

$= 0.7^{15} + \binom{15}{1} \cdot 0.3^{1} \cdot 0.7^{14} + \dots \binom{15}{3} 0.3^{3} \cdot 0.7^{15-3}$

$= 0.297$

b) $P(X \geq 2) = P(x=2) + \dots + P(X=15)$

$1 - P(x=0) - P(x=1) = 0.965$

Geometric Distribution

The geometric distribution is a discrete Probability distribution that models the number of trials needed to achieve the first Success

One parameter is θ, where θ is Probability of Success in each trial.

PMF: $P(X=x) = (1-p)^{x-1} \cdot p$

The first $x-1$ trials must be failures times the Probability of Success. p is Prob of Success

$E[x] = \frac{1}{\theta}$ $Var(x) = \frac{1-\theta}{\theta^2}$

Distribution	Models
Bernouli	Single trial
Binomial	Number of successes in n trials
Geometric	Number of trials to first Success.

Problem 3.3

a) Let x = # of trials untill 6 appears

$x \sim Geo(\theta = \frac{1}{6})$

$P(x=3) = \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) = \frac{25}{216}$

If its a 6 on the third toss, had to have been 1-5 on first two.

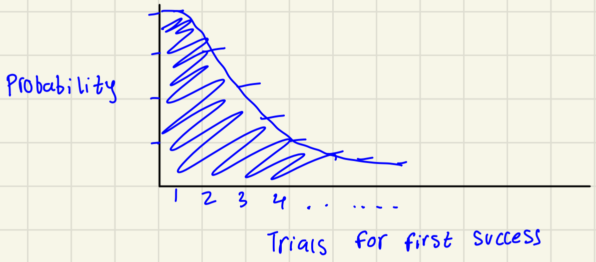
b) $P(x \geq 3) = 1 - P(x=1) - P(x=2)$

$= 1 - \frac{1}{6} - \left(\frac{5}{6}\right) \cdot \frac{1}{6} = \frac{25}{36}$

$P(x \leq 3) = P(x=1) + P(X=2) + P(x=3)$

$= (1-0.3)^{1-1} \cdot 0.3 + (1-0.3)^{1} \cdot 0.3 + (1-0.3)^{2} \cdot 0.3$

$= 0.657$



The first trial always has highest probability at $X=1$.

For Geometric Distribution:

Can also use the CDF: $F(x) = P(X \leq x) = 1 - (1-p)^x$

Problem 3.4

The memoryless property of the geometric distribution, is a unique characteristic where probability of needing more trials for a success, independent from how many have been completed

a) Show that for a positive integer a,

$$P(x>a) = q^a, \text{ where } q = 1-p$$

Failure : $q = 1-p$

Event $X>a$ means that the first a trials are all failures.

$$P(x>a) = (1-p)^a = q^a$$

b) $P(X>a+b \mid x>a) = P(x>b)$

By the Definition of Conditional Probability:

$$P(x>a+b \mid x>a) = \frac{P(x>a+b \cap x>a)}{P(x>a)}$$

Because $P(x>a+b)$ implies $P(x>a)$ the event can just be

$$= \frac{P(x>a+b)}{P(x>a)}$$

From a) $P(x>a) = q^n$ and $P(x>a+b) = q^{a+b}$

$$= \frac{q^{a+b}}{q^a} = q^b = P(x>b)$$

Negative Binomial Distribution

Models the number of trials needed to achieve a specified number of successes in a sequence of independent and identically distributed Bernoulli trials.

Formula:

If X reperesents the number of trials needed to get r successes, X follows negative Binomial distribution

$$PMF = P(X=k) = \binom{k-1}{r-1} \cdot p^r (1-p)^{k-r}$$

- p is the probability of success
- 1-p prob of failure
- r is # of required successes
- k is the total number of trials, needed for r success
- $\binom{k-1}{r-1}$ Binomial Coeff, counts ways to arrange r-1 success in k-1 trials.

Expected value: $E[x] = \frac{r}{p}$

Variance: $Var(x) = \frac{r(1-p)}{p^2}$

Cool Note:

When r=1, negative binomial distribution becomes geometric

The first x-1 trials must result in r-1 successes.

Problem 3.5

a) $P = 0.3$
$r = 2$
$X = 10$

$$P(X=K) = \binom{K-1}{r-1} \cdot p^r (1-p)^{K-r}$$

$$P(X=10) = \binom{10-1}{2-1} \cdot 0.3^2 (1-0.3)^8$$

$$P(X=10) = \binom{9}{1} \cdot 0.3^2 (0.7)^8$$

$$P(X=10) = 0.0467$$

b) $P(X \leq 4) = P(X=2) + P(X=3) + P(X=4)$

$$= 0.3^2 + \binom{3-1}{2-1} \cdot 0.3^2 \cdot \boxed{0.7^1} + \binom{4-1}{2-1} \cdot 0.3^2 \cdot \boxed{0.7^2}$$

$$= 0.348$$

This term is introduced because we have 1 and then 2 failures within the first x trials.

Where $r=2$, X = changes, $P=0.3$

Problem 3.6

$P = 0.5$ $F = 0.5$
$x = 4, 5, 6, 7$
$r = 4$
4 wins means series is over
Probability: one team wins
4 in arow is $P(\text{sweep}) = 0.5^4$
$P(\text{Sweep}) = \frac{1}{16}$

Prob of 5 games: $P(x=5) = \binom{4}{3} \cdot 0.5^4 \cdot 0.5 = 4 \cdot \frac{1}{32} = \frac{1}{8}$

Prob of 6 games: $P(X=6) = \binom{5}{3} \cdot 0.5^4 \cdot 0.5^2 = 10 \cdot \frac{1}{64} = \frac{10}{64} = \frac{5}{32}$

Prob of 7 games: $p(x=7) = \binom{6}{3} \cdot 0.5^4 \cdot 0.5^3 = 20 \cdot \frac{1}{128} = \frac{5}{32}$

Pmf Let Y be # of games till one team wins.

$y =$	4	5	6	7
$f(y) =$	0.125	0.25	0.3125	0.3125

Multiply all Probabilities by 2.

Problem 3.7

Series length	4	5	6	7
Obsered Freq	17	15	16	12
Observed prob:	$\frac{17}{60} = 0.283$	$\frac{15}{60} = 0.25$	$\frac{16}{60} = 0.267$	$\frac{12}{60} = 0.20$

NO, they dont fit a similiar distribution, teams ussually are not evenly matched up in the finals.

Hypergeometric Distribution

Probability distribution that describes the likelihood of getting a certain number of successes in a sequence of draws from a finite population without replacement.

Without Replacement - Impies that probabilities will change based on draws, making the draws dependent.

Key Characteristics

- Finite population, divided into successes and failures
- Draw x items without replacement.
- Drawing specific number of success in given amt of tries

$$\text{PMF: } P(X=M) = \frac{\overbrace{\binom{M}{x}}^{\text{ways to choose}}\binom{N-M}{n-x}}{\binom{N}{n}}$$

← ways to choose the "rest"

→ Total outcomes

Where:

- N is total size of population
- M is total "Successess" in population
- n is the number of draws
- X is random var reperesenting the # of success that we want.

$$\text{Mean: } \mu = \frac{nM}{N}$$

$$\text{Var}(x) = \frac{nM(N-M)(N-n)}{N^2(N-1)}$$

The term $\frac{N-n}{N-1}$ is called the finite Population adjustment.

Note: If sampling less than 5% of the total population, the hypergeometric and binomial distribution will give "close" answers.

Problem 3.8

Box contains : 8 red balls
2 black balls

a) Hypergeometric :

$$P(4\text{ Red}) = \frac{\binom{8}{4}\cdot\binom{2}{1}}{\binom{10}{5}} = \frac{5}{9}$$

b) $P(R) = 0.8$
$P(B) = 0.2$

Total outcomes :

$$\text{Bin} \sim \underbrace{\binom{5}{4}}_{\text{Choosing 4/5 Red}} \cdot \underbrace{(0.8)^4}_{\text{Prob of choosing reds}} \cdot \underbrace{0.2}_{\text{leftover from the 1 black}}$$

works with geo and neg binomral

* c) $f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$

$p = 0.8$
$x = 4$
$k = 1$

$$f(x=4) = \binom{4-1}{1-1} \cdot 0.8^1 (1-0.8)^{4-1}$$
$$= \binom{3}{0} \cdot 0.8^1 (0.2)^3$$
$$= \frac{4}{625}$$

Problem 3.9

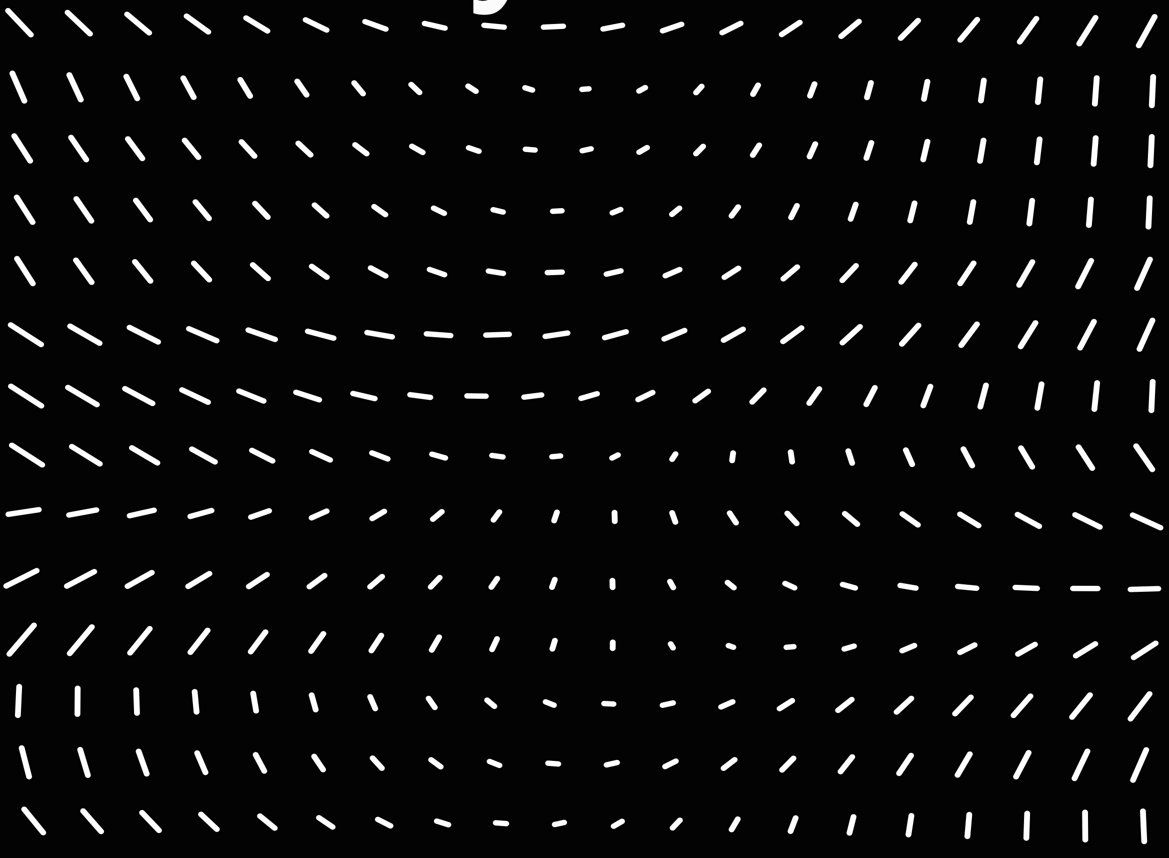
120 applicants
80 qualified
40 not qualified

a) $$\frac{\binom{80}{2}\cdot\binom{40}{3}}{\binom{120}{5}} = 0.164$$

b) $p = \frac{2}{3}$

$$P(x=2) = \binom{5}{2}\left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^3 = \frac{40}{243} = 0.1646$$

Final Exam Study Guide



Topic 1: Probability

Important Terms and Definitions

- Classical prob. often defined through relative frequency of occurance of a random event
- Sample space (S) is the set of all possible outcomes of an experiment
- Random Event (A, B) is a subset of the sample space

Rules of Probability

Multiplicative Rules: For 2 independent events, the probability of both happening $P(A \cup B)$ is $P(A) \cdot P(B)$

Addition Rule: For 2 mutually exclusive the probability of doing event A or doing event B. $P(A \cap B) = P(A) + P(B)$

↳ Non mutually exclusive
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

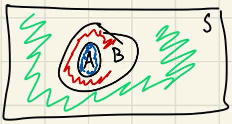
Counting Rules

- Permutations, Order Matters: The # of ways to arrange r objects, selected from n distinct objects. $_nP_r = \frac{n!}{r!(n-r)!}$ or $5p3 = 60$

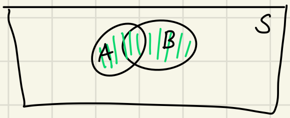
- Combinations, Order Does NOT Matter: The # of ways to choose r subjects from n distinct objects. $_nC_r = \binom{n}{r}$ or $5c3$

Review of set notation

$A \subset B$ - A is a subset of B

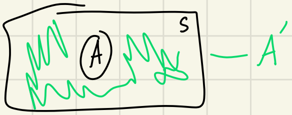


$A \cup B$ union of A and B



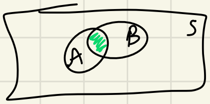
A or B
or Both

A' or $\bar{A}$: The complement of A.



$A \cap B$ Intersection of A and B

Both A and B



$(A \cap B)' = A' \cup B'$

$(A \cup B)' = A' \cup B'$

Probability Properties

1. $P(A) \geq 0$
2. $P(S) = 1$
3. If A_1, A_2, A_3 are pairwise mutually exclusive, then the sum of their union is all of their individual probabilities summed up.
   $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

Sample Point Method

Thm. 2.2 An experiment can give n different outcomes, each outcome is equally likely. Event A consists of m of these out comes.

Then $P(A) = \frac{m}{n} = \frac{\#\text{ of sample pts in } A}{\#\ \ldots\ \text{in } S}$

Common Rules

- $P(A') = 1 - P(A)$
- Independent Events: $P(A \cap B) = P(A) \cdot P(B)$
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
- De Morgan's Law: $A' \cap B' = (A \cup B)'$ and $(A \cap B)' = A' \cup B'$

Conditional Events:

- Probability of A given B:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(B) \cdot P(A \mid B)$$

Mutually Exclusive

Def: 2 event's can not occurr at the same time.

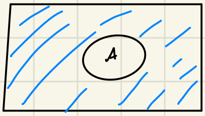
$$P(A \cap B) = 0$$

ote: Mutually exclusive events can not be independent.

Because the happening of one event directly influences, the other event.

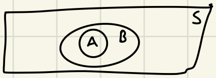
Rules of Probability

1.) $P(A') = 1 - P(A) \iff P(A') = 1 - P(A)$



2.) $P(\emptyset) = 0$ for any sample space S.

3.) A, B are events in S. and $A \subset B$.
Then $P(A) \leq P(B)$



4.) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A, B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

Conditional Probability

If A and B are two events in the sample space S, and $P(A) \neq 0$

The conditional probability of B given A:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Law of Probability

If A, B are two events in S, $P(A) \neq 0$.

Then: $P(A \cap B) = P(B) \cdot P(A|B)$

which is just

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ rearranged.

Independent Events

Def: Two events A, B are independent if
$(A \cap B) = P(A) \cdot P(B)$

Being independent means event B does not influence the probability of event A.

Mutually Exclusive

Def: 2 event's can not occur at the same time.

$$P(A \cap B) = 0$$

Note: Mutually exclusive events can not be independent.

Because the happening of one event directly influences, the other event.

Mutual Independence

Def: A, B, C are mutually independent if:

$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ ✓
$P(A \cap B) = P(A) \cdot P(B)$ ✓
$P(A \cap C) = P(A) \cdot P(C)$ ✓
$P(B \cap C) = P(B) \cdot P(C)$ ✓

Pairwise Independence

A, B, C are pairwise indep if:

$P(A \cap B) = P(A) \cdot P(B)$ ✓
$P(A \cap C) = P(A) \cdot P(C)$ ✓
$P(B \cap C) = P(B) \cdot P(C)$ ✓

Partition - a bunch of cases that cover all possible outcomes.

Thm 2.12 (Law of Total Probability)

If events $B_1, B_2 \ldots. B_K$ form all sample points in S, and they are mutually exclusive and 1 most occur.

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i)$$

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) \ldots. P(B_K) \cdot P(A|B_K)$$

Bays Theorem

Bay's Theorem is a way of updating Probabilities based on new information.

$$P(B_i|A)=\frac{P(A|B_i)\cdot P(B_i)}{P(A)}$$

- where $P(A|B_i)$ is the Probability of A happening given B_i (likilihood)
- $P(B_i)$ is the old Probability
- $P(A)$ is the total Probability

Baye's Theorem is directly related to Law of total Probability:

$$P(B_i|A)=\frac{P(A|B_i)\cdot P(B_i)}{P(A)}$$

$$P(A)=P(B_1)\cdot P(A|B_1)+P(B_2)\cdot P(A|B_2)\ldots\ldots P(B_K)\cdot P(A|B_K)$$

→ Longer version Could be

$$P(B_i|A)=\frac{P(A|B_i)\cdot P(B_i)}{P(B_1)\cdot P(A|B_1)+P(B_2)\cdot P(A|B_2)\ldots\ldots \text{ to } B_K}$$

"With Replacement Questions"

Permutations with replacement from n^K.
Where n is # of possible outcomes and K is how many are Chosen (how many positions).

Combinations with replacements $C(n+K-1,K)$
Where n is # of options and K is how many your Choosing:

Topic 1: Practise

1.4)

a) 5c2

Ranked: $\{1,2,3,4,5\}$

Total possibilities is 5c2

$(1,4)\ (1,5) = \frac{2}{5c2}$

b) (1,2) (1,3) (1,4), (1,5)
(2,3) (2,4) (2,5) $\frac{7}{5c2}$

Problem 1.5)

Total outcomes: 52c5

13 ranks
4 suits

To pick 3 of the same rank,

$$\frac{\binom{13}{1}\cdot\binom{4}{3}\cdot\binom{12}{1}\cdot\binom{4}{2}}{52c5}$$

$$\frac{\binom{13}{1}\cdot\binom{4}{1}}{52c13}$$

C-GPt

1) 10 Total

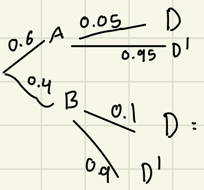
a) $= \frac{\binom{5}{3}}{10c3}$

b) $\frac{\binom{5}{1}\cdot\binom{3}{1}\cdot\binom{2}{1}}{10c3}$

2) 2 Dices rolled
6^2 outcomes
Sample Space: 36
A = Sum is 7

(1,2) (1,3) (1,4) (1,5) (1,6)
(2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

Com · to: 1.5 b)



$$P(A|D) = \frac{P(D|A)\cdot P(A)}{P(D)}$$

$P(D) =$
$P(D) = 0.04 + 0.03 = 0.07$
$P(A) = 0.6$
$P(D|A) = 0.05$

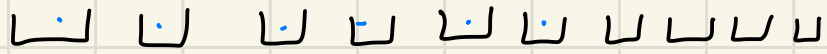
$P(A|D) = \frac{3}{7}$

0.04 + 0.03

Total: 15c5

$$\frac{\binom{8}{2}\cdot\binom{7}{3}}{15c5}$$

1) 6 Book Placed in 10 boxes



P(at least 1 box has more than 1 Book) =
= 1 − P(each book gets its own box)

# of ways to assign 6 objects to n objects, 10 p6 = 151200

Each book has : 10^6 options

$$1 - \underbrace{\frac{10P6}{10^6}}_{\text{Each book gets its own box.}} = 0.8488$$

b) $P(B1 \cap B2) = P(B_1) + P(B2) - P(B1 \cup B2)$

Box 1 and box 2 should have the same Prob of being empty:

9 to Choose, if box 1 empty: $\frac{9P6}{9^6} = 0.1138$

If Both are empty: $\frac{8P6}{8^6} = 0.077$

$P(B1 \cap B2) = 0.1138 \cdot 2 - 0.077 = 0.1506$

Actual:

$\frac{9^6}{10^6} = 0.53$

P(both empty) $= \frac{8^6}{10^6} = 0.262$

P(one Empty) $= 0.53 + 0.53 - 2(0.262)$
$= 0.536$

Revisit 1.6 / 1.7

Topic 2: Probability Distributions, Densities and Expectations

Discrete Random Variable - Can take on a countable number of distinct values

Probability Mass Function (PMF): $P(X=x)$ gives the probability that the discrete random variable X is equal to a specific value x.

Formula: $P(X=x) = p(x)$

Where $p(x)$ is the probability that the random variable X takes on the value of x. The sum of all possibilities must equal 1: $\sum P(X=x) = 1$

The variable X is essencially a function defined on S, that maps the outcomes of experiments to numbers.

Continous Random Variable - take values in some intervals of real #s.

Probabity Distributions for Discrete R.V's

Def: If x is a discrete r.v. Then,

$f(x) = p(X=x)$ for each x in the range of X is called the pmf of X.

Cumulative Distribution Function (CDF)

Gives the probability that a random variable X is less than or equal to a certain value.

$$F(x) = P(X \leq x)$$

This means for a value x, CDF gives the probability that X will take on a value less than or equal to x.

CDF:
$$F_X(x) = \begin{cases} 0 & \text{for } x < -3 \\ \frac{1}{8} & \text{for } -3 \leq x < -1 \\ 0.5 & \text{for } -1 \leq x < 1 \\ \frac{7}{8} & \text{for } 1 \leq x < 3 \\ 1 & \text{for } x \geq 3 \end{cases}$$

Frequency Distribution for Observed Data

Observed Data - data one has seen

Frequency how many times that number has occurred

Types of Frequency

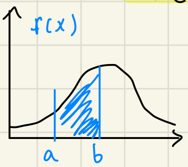
Absolute Freq - Actual number of times a value occurs

Relative Freq - $\dfrac{\text{Absolute Freq}}{\text{Total \# of Data pts}}$

Continous R.V, PDF and CDF.

Def. For a continous random variable X, the PDF f(x) gives the Probability that X falls within a certain range. The probability that x lies between a and b is:

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$



area under the graph.

Note: ① If x is cont r.v then $P(X=x)=0$

② The total area under the pdf curve is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The Cumulative Distribution Function for Continous R.V

Def: The CDF for continous R.V X, F(x) gives the probability X takes a value less than or equal to x

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)(t)\,dt \text{ , for } -\infty < x < \infty$$

Common Question
Find the CDF, given PDF.

Relationship between CDF and PDF

- The CDF is derived by integrating the PDF
- The PDF is found by differentiating the CDF

$$f_X(x) = \frac{d}{dx} F_X(x)$$

- $F(-\infty)=0$, $F(\infty)=1$

PDF → CDF
Integrate

CDF → PDF
Derivative

Mathematical Expectations

Expected Value: The long run average value of a R.V, if you were to do an experiment many times.

For Discrete RV: Sum all possible outcomes, weighted by probability

$$E(X) = \sum_{x} x \cdot P(X=x)$$

For continous RV: Integrate the values, weighted by densities

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Problem 2.11

a)

x:	1	-1
$f(x)$:	$1-\left(\frac{5}{6}\right)^4$	$\left(\frac{5}{6}\right)^4$
	0.518	

$$P(X=-1) = \left(\frac{5}{6}\right)^4$$

$$E(x) = 1 \cdot 0.518 + (-1)\left(\frac{5}{6}\right)^4$$

$$= 0.036 \qquad > 0$$

∴ So he is expected to win a bit.

Properties of Expectations

1. $E(b) = b$, The expectation of a constant is simply the constant

2. $E(aX+bY) = aE(X) + bE(Y)$

If X and Y are r.v and a, b are constants, you can split them up.

3. $E(X+Y) = E(x) + E(Y)$

The expected value of a sum is sum of expected values

$$E[ax+b] = a\,E[X] + b$$

$$V[ax+b] = a^2\sigma^2$$

Problem 2.13

J, Q = \$15
K, A = \$5
2,3,4,5,6,7,8,9,10 = −\$4

X = money he makes from 1 draw

x:	15	5	-4
$f(x)$:	$\frac{8}{52}$	$\frac{8}{52}$	$\frac{36}{52}$

Discrete

$$E(X) = 15 \cdot \frac{8}{52} + 5 \cdot \frac{8}{52} + (-4)\frac{36}{52}$$

$$E(x) = \$0.31$$

Note: This is a discrete random variable, this means he is expected to win 31¢ on avg.

Problem 2.14

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)}\, dx \longrightarrow \text{how to integrate}$$

$$= \frac{2}{\pi} \ln(1+x^2) \Big|_{x=0}^{1}$$

$$= \frac{\ln 4}{\pi}$$

$$= 0.4413$$

how to integrate

$$U = 1+x^2$$

$$dU = 2x$$

$$\int \frac{2\,dU}{\pi \cdot U}$$

$$= \frac{2}{\pi} \ln U$$

Theorem 4.6

$$V(X) = E(X^2) - \boxed{[E(X)]^2}$$

μ is a constant

$$= E[(X-\mu)^2]$$
$$= E[x^2 - 2\mu x + \mu^2]$$
$$= E(x^2) - E[2\mu x] + E(\mu)^2$$
$$= E(x^2) - 2\mu E(x) + \mu^2$$
$$= E(x^2) - 2\mu^2 + \mu^2$$
$$= E(x^2) - \mu^2$$

Notes:

1.) $V(X) \geq 0$

2.) $E(X^2) \geq [E(x)]^2$

3.) $V(aX+b) = a^2 \cdot V(x)$ for any constant a, b

Chebyshev's Theorem

Provides a way to estimate the probability that a random variable lies within a certain number of standard deviations from the mean.

For any random variable X with mean μ and variance σ^2, for any $k > 1$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

or

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

When to use either?

k is the number of std's away it is from mean.

Note: This theorem provides a lower bound for the prob that X takes values within k std of the mean

Notes on Chebyshev's Thm

- Applicable on any type of distribution
- Giveaway to use it, is the two bounds given are equal distant apart

$$K = \frac{\text{Bounds} - \mu}{\sigma}$$

K – # of Standard deviations.

$\frac{1}{k^2}$ is the upper bound

If it asks within kσ of the mean, you use $1 - \frac{1}{k^2}$. Otherwise, if it is outside the mean use $\frac{1}{k^2}$.

Theorem 4.1

- Expectation of a function of a r.V

If x is a discrete r.V with pmf $f(x)$,

$$E[g(x)] = \sum_{\text{all } x} g(x) \cdot f(x)$$

So, we apply some function g to X before taking Expectation

- If X is a con't r.v with pdf $f(x)$, then $E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f(x) \cdot dx$

Problem 2.15

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $E[X] = \int_0^1 x \cdot 2x\, dx = \frac{2x^3}{3}\Big|_0^1$

$= \frac{2}{3}$

b) $Y = 200X^{\frac{1}{2}} + 60$

$E[Y(X)] = \int_0^1 (200X^{\frac{1}{2}} + 60) \cdot 2x\, dx$

$= \$220$

c) $P(Y > 180) = P(200x^{\frac{1}{2}} + 60 > 180)$

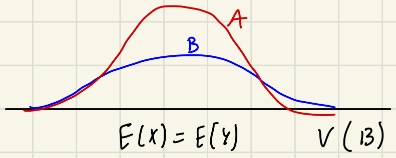
$= P(200x^{\frac{1}{2}} > 120)$

$= P\left(x^{\frac{1}{2}} > \frac{120}{200}\right)$

$= P(X > (0.6)^2)$

$= \int_{0.36}^{1} 2x\, dx = x^2\Big|_{0.36}^{1}$

$= 0.8724$



Moments of a Random Variable

- a moment is a way to describe various aspects of a distribution of r.V X.
- The r'th moment of X is $\mu_r' = E(X^r)$
- Moments are used to understand shape and characteristics of distributions.

Ex

First Moment $(r=1)$: $E(X)$, the mean of X.

Second Moment $(r=2)$. $E(X^2)$

Central Moments:

The r-th central moment is $\mu_r = E[(X-\mu)^r]$

also called r'th moment about the mean.

Variance and Standard Deviation

The Variance is a measure of how much X deviates from it's mean μ.

This is the averaged square distance of X from it's mean.

- $Var(X) = E[(X-\mu)^2]$

$V(X) = \sigma^2$

$\sigma = \sqrt{V(X)}$: is called standard deviation of X. Denoted by σ.

Moment Generating Function

Recall that $\mu_r' = E(x^r)$, $r=1,2,3$

If two random variables X and Y have the same r^{th} moments for all $r=1,2,3$ then X and Y have the same probability distribution.

Discrete random variable X with pdf $f(x)$ the mgf is defined as $M_X(t) = E[e^{tx}] = \sum_{\text{all } x} e^{tx} \cdot f(x)$

For Continous R.V:

$$M_x(T) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

The moment generating function allows us to find all the different moments ($E[x^2], E[x^{53}]$ etc)

Finding the first derivative of the MGF and setting it to 0 gives you First Moment

Second Derivative Set to 0, gives the Second moment.

First Moment: $\left.\frac{dm(t)}{dt}\right|_{t=0} = E(x)$ —— Set First derivative, Plug in 0

second moment $\left.\frac{d^2m(t)}{dt^2}\right|_{t=0} = E(x^2)$ —— Set second derivative, plug in 0

In general,

$$\left.\frac{d^r \cdot m(t)}{dt^r}\right|_{t=0} = E(x^r) = \mu_r'$$

---

problem 2.21

$$f(x) = \frac{\binom{5}{x}\binom{3}{2-x}}{\binom{8}{2}} \quad \text{for } x=0,1,2$$

Giveaway its Discrete

1) Find Probabilities by Plugging in x values.

2) Multiply e^{tx} times the probability of x's.

a) $M_X(t) = E(e^{tx}) = \sum e^{tx} \cdot f(x)$

Pmf:

x :	0	1	2
$f(x)$:	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

$= e^{t\cdot 0}\cdot f(0) + e^{t\cdot 1}\cdot f(1) + e^{t\cdot 2}\cdot f(2)$

$= \frac{3}{28} + \frac{15}{28}\cdot e^{t} + \frac{5}{14} e^{2t}$

b) Finding $E(X)$ and $V(X)$ through the mgf.

set $d\,m(t) = 0$

$E(x) = \left.\frac{dm(t)}{dt}\right|_{t=0} = \frac{15}{28}e^{t} + \frac{5}{14}\cdot 2e^{2t}$

$= \frac{15}{28} + \frac{10}{14} = \frac{35}{28} = \frac{5}{4}$

$E(x) = \frac{5}{4}$

Second derivative:

$= \left.\frac{15}{28}e^{t} + 4e^{2t}\cdot\frac{5}{14}\right|_{t=0}$

$= \frac{15}{28} + \frac{10}{7}$

$E(x^2) = \frac{55}{28}$

$V(x) = E(x^2) - [E(x)]^2$

$V(x) = \frac{55}{28} - \left(\frac{5}{4}\right)^2$

$V(x) = \frac{45}{112}$

Problem 2.23

$M(t) = \frac{1}{6}\boxed{} + \frac{2}{6}e^{\boxed{2}t} + \frac{3}{6}e^{\boxed{3}t}$

$M_X(t) = E[e^{tx}] \longrightarrow$ By Definition

PMF is directy related to the MGF.

The random variables take on the values of 0, 2, 3 because these are the coeffecient on the t terms.

The terms suggest: $P(X=0) = \frac{1}{6}$

$P(X=2) = \frac{2}{6}$

$P(X=3) = \frac{3}{6}$

The pmf gives you probabilities directly

The mgf reperesents these probabilities in a transformed way.

How to Solve MGF Questions

1) Determine if you are given a continous r.V or discrete r.v. (Continous means PDF, Discreate means PMF). Then use either $\sum_{K}^{\infty} e^{tx} \cdot f(x)$ or $\int_{-\infty}^{\infty} e^{tx} \cdot f(x)$.

2) Factor exponential terms/ simplify
3) Solve integral
4) Look at behaviour to see how, we can make f(x) a finite value.
5) Plug in limit, solve
6) Find First/Second moment from derivatives.

Exponent Rules

1) $e^{a} \cdot e^{b} = e^{a+b}$

2) $e^{a \cdot b x} = e^{(ab)x}$

$(e^{2x} \cdot e^{-3x}) = e^{(2-3)x} = e^{-x}$

Key Properties for Exponentials

- For e^{ax}, if $a > 0$, e^{ax} grows toward infinity as $x \to \infty$
  if $a < 0$, e^{ax} decays toward 0 as $x \to \infty$

Behavior at infinity:

$a < 0$, $e^{ax} \to 0$ as $x \to \infty$

$a > 0$, $e^{ax} \to \infty$ as $x \to \infty$

How to Solve MGF Questions, Discrete Geometric Series

1) recognize discrete case
2) Simplify and get in correct form
3) Apply geometric series, by plugging in smallest value for K, then use this as numerator for geometric series
4) Plug back in and find valid bounds ($|r| < 1$)
5) Get Mgf and compute $E[X]$, $E[X^2]$, $V[X]$.

$$f(x) = \begin{cases} 3e^{-3x}, & \text{if } x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$M_{TX} = \int_{-\infty}^{\infty} e^{tx} \cdot 3e^{-3x}$$

$$M_{TX} = 3\int_{0}^{\infty} e^{tx} \cdot e^{-3x}$$

$$= 3\int_{0}^{\infty} e^{x(t-3)} \longrightarrow \text{need to make sure } t-3 < 0$$

$$t < -3$$

$$= \frac{3 \cdot e^{x(t-3)}}{t-3} \Big|_{0}^{\infty} = \frac{3}{t-3} \text{ for } t < -3$$

$$M_{TX} = \frac{3}{t-3} \text{ when } t < 3$$

$$f'(x) = 3\,(t-3)^{-1}$$

$$= 3 \cdot (-1)(t-3)^{-2}$$

$$= \frac{-3}{(t-3)^2} \Big|_0 = \frac{-3}{9} = -\frac{1}{3}$$

$$f''(x) = (-3)(2)(t-3)^3$$

$$= \frac{-6}{(t-3)^3} \Big|_0 = \frac{2}{9} = \frac{1}{9}$$

$$=$$

$$M_{TX} = \sum_{K=0}^{\infty} e^{tx} \cdot \frac{1}{2^x}$$

$$= \sum_{K=1}^{\infty} \left(\frac{e^t}{2}\right)^x = \frac{\frac{e^t}{2}}{1 - \frac{e^t}{2}} = \frac{e^t}{2-e^t}$$

$$f'(x) = \frac{e^t}{2-e^t}$$

$$= \frac{e^t(2-e^t) - (e^t)(-e^t)}{(2-e^t)^2}$$

$$=$$

$$= \frac{e^t\left[(2-e^t)(-e^t)\right]}{(2-e^t)^2}$$

$$= \frac{2e^t}{(2-e^t)^2} \Big|_0 = 2$$

Topic 3 - Discrete Random Variables and their distributions

Bernoulli Distribution

Definition: A bernoulli trial is an experiment with 2 mutually exclusive distinct outcomes. There are Successes (P) and Failures (1-P).

Example: Flipping a coin

PMF: $P(X=x) = p^{x}(1-p)^{1-x}$, where x is 0, 1 (0 is failure, 1 is success)

Mean: $\mu = p$

Variance: $\sigma^2 = p(1-p)$

Binomial Distribution

- Repeat independent Bernoulli trials with n finite times.

PMF: $P(X=k) = \underbrace{\binom{n}{k} p^{k}}_{\text{\# of success}} \underbrace{(1-p)^{n-k}}_{\text{\# of failures}}$, $k = 0, 1 \dots n \longrightarrow$ K is the # of Successes

Mean: $\mu = np$

Variance: $\sigma^2 = np(1-p)$

Note: 1) Probability of Success is P
2) Finite number of trials
3) Same mean and variance as bernoulli, just multiplied by n.
4) Bernoulli distribution is a special case of Binomial dist, when n = 1

Geometric Distribution

Def: Counting the number of trials Until the first Success.

PMF: $P(X=k) = \underbrace{(1-p)^{k-1}}_{\text{\# of failures}} \cdot \underbrace{p}_{\text{First success}}$, where p is Probability of success and K is # of trials to first success.

Mean: $\mu = \frac{1}{p}$

Variance: $\sigma^2 = \frac{1-p}{p^2}$

The first trial always has highest Probability at X = 1.

Negative Binomial Distribution

Models the number of trials needed to achieve "r" successes, for Bernoulli events

$$\text{PMF: } P(X=k) = \binom{x-1}{k-1}\underbrace{p^{k}}_{\text{Success}}\underbrace{(1-p)^{x-k}}_{\text{Failure}}$$

, where P is Probability of Success

, where x is the number of trials, needed for r successes

, where k is the required amount of successes.

$$\text{Mean: } \mu = \frac{r}{p}$$

$$\text{Variance: } \sigma^2 = \frac{r(1-p)}{p^2}$$

Cool Note:

When r=1, negative binomial distribution becomes geometric

Hypergeometric Distribution

Probability distribution that describes the likelihood of getting a certain x number of successes in a sequence of draws from a finite population without replacement.

Without Replacement - Impies that Probabilities will change based on draws, making the draws dependent.

Revisit: Hypergeo -vs. Binomial "without" vs "with" replacement questions.

Key Characteristics

- Finite population, divided into successes and failures
- Draw x items without replacement.
- Drawing specific number of success in given amt of tries

$$\text{PMF: } P(X=M) = \frac{\overset{\text{ways to choose}}{\binom{M}{x}}\binom{N-M}{n-x}}{\binom{N}{n}}$$

← ways to choose the "rest"

⟶ Total outcomes

$$\text{Mean: } N = \frac{n \cdot M}{N}$$

$$\text{Variance: } \sigma^2 = \frac{nM(N-M)(N-n)}{N^2(N-1)}$$

Where:

- N is total size of Population
- M is total "successess" in population
- n is the number of draws
- X is random var reperesenting the # of success that we want.

Poisson Distribution

The Poisson distribution models the number of rare events that occur in a fixed interval of time or space.

The events must be:

1.) Randomly and Independently of each other
2.) At a constant average rate (λ) (number of expected outcomes)
3.) Events can not occur simultaneously

PMF: $P(X=k) = \dfrac{\lambda^{k} \cdot e^{-\lambda}}{k!}, k = 0, 1, 2$ → where λ = average rate of event
where k = number of events in interval

Mean: $\mu = \lambda$

Var: $\sigma^2 = \lambda$

Calculator Tip

- Poisson PD for exact: $P(x=4), \lambda=5$
- Poisson CD for $X \leq k$: $P(x \leq 1) = P(x=0) + P(x=1)$

Note: Binomial → Poisson Distribution approx, when n is large and when p is small and $np \approx \lambda$.

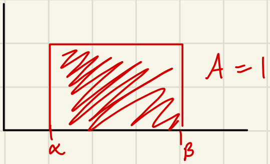
Topic 4: Continous Random Variables and their Distributions

Reminder: values come from a continous range
The probability of a specific value is 0, but over a positive interval is $P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$.

Uniform Distribution

The uniform distribution models situations where every outcome in a certain interval is equally likely.

PDF: $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & , \text{if } \alpha < x < \beta \\ 0 & , \text{elsewhere} \end{cases}$



- Height of PDF is constant because outcomes are equally likely.
- Area under curve is 1

Mean: $\mu = \frac{a+b}{2}$

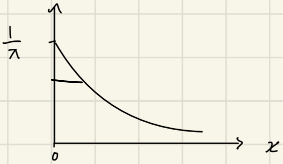
$P(c \leq X \leq d) = \frac{d-c}{b-a}$ } CDF, which can be used to compute probability.

Variance: $\sigma^2 = \frac{1}{\lambda^2}$

The Exponential Distribution

Def: Modeling the time until the next event in Poisson process.

- Events occur independently
- Average rate of occurrence is constant



PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0, & x < 0 \end{cases}$$

where λ is the rate parameter: number of events per unit time
$f(x)$ decays exponentialy as x increases.

Mean: $\mu = \frac{1}{\lambda}$

$V(X)$: $\sigma^2 = \frac{1}{\lambda^2}$

Probability of Intervals

$P(X \leq t) = 1 - e^{-\lambda t}$
$P(X > t) = e^{-\lambda t}$

Exponential Distribution - Continued

Relationship Poisson and Exponential Distribution

Poisson describes chances of event occuring. Exponential gives distance between events that are described by Poisson situation.

Solving Problems with Exponential Distribution

1.) Find λ: $\lambda = \frac{\#\text{ of events}}{\text{Total time}}$

2.) Use PDF for exact probabilities:

$P(X = x)$, $f(x) = \lambda e^{-\lambda x}$

3.) Use CDF

To find $P(X \leq x)$: $F(x) = 1 - e^{-\lambda x}$

To find $P(X > x)$: use $1 - F(x) = e^{-\lambda x}$

Memoryless Property

$$P(X > s + t \mid X > s) = P(X > t)$$

The probability of an event occurring in the future does not depend on how much time has already passed.

Reminder By Definition.

- $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$
- $f_X(x)$ is the derivative of $F_X(x)$
- Expectation: $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$
- Variance: $E[X^2] - [E[X]]^2$

Gamma Distribution

Def: Models the sum of independent exponential random variables.
This distribution models wait times for $r > 1$ events.

The Gamma Distribution is parameterized:

- Shape Parameter $(\alpha > 0)$: Reperesents the number of events.
- Rate Parameter $(\beta > 0)$: Reperesents the rate of process. (Scale)
- $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt$

Key Properties:

Mean: $E[X] = \alpha\beta$
Variance: $Var[x] = \alpha\beta^2$

Note: When $\alpha = 1$, the gamma distribution reduces to the exponential distribution

Example: Waiting for the 3rd event in a Poisson Process.

Note: The gamma distribution models how long you'll wait for multiple events to occur. As r increases, it becomes more symmetric, resembling normal distribution.

Beta Distribution

The beta distribution models random variables bound between 0 and 1.
A random variable X follows a Beta distribution if:

$$f_X(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 \leq x \leq 1$$

Where α, β are shape parameters.

Mean: $\frac{\alpha}{\alpha+\beta}$

Variance: $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

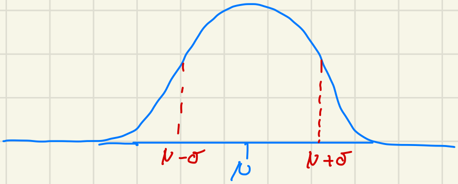
Poisson - # of rare events or prob of that event occuring

exponential - time in between Poisson events, or till next Poisson event

Gamma - time it takes to achieve "r" number of exponential events

The Normal Distribution

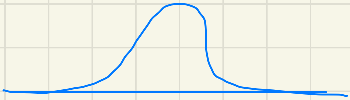
- This is a probability distribution that is symmetric about it's mean.
- The shape of it's PDF is bell curved



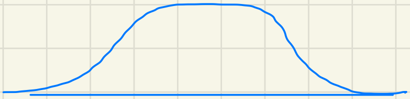
Key Characteristics

- Mean (μ): Determines centre of distribution
- Standard Deviation (σ): Determines how spread the curve is.

If σ is smaller:



If σ is larger:



PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

The normal distribution is asymptotic and never actually touches 0.

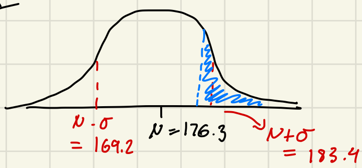
Emperical Rule

- 68% of data falls within 1 standard deviation ($\mu \pm \sigma$)
- 95% of data falls within 2 standard deviations ($\mu \pm 2\sigma$)
- 99.7% of data falls within 3 standard deviations ($\mu \pm 3\sigma$)

The Standard Normal Distribution

- Is when a normal distribution with $\mu = 0$ and $\sigma^2 = 1$, $\sigma = 1$

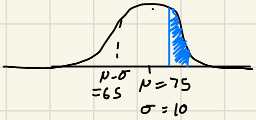
EX



Normal CD: $P = 0.3$

To find 90th Percentile, first Standardize: $z = \frac{x-\mu}{\sigma}$

Problem 4.9



a) Find between 80 and 90

Normal CD:

$P = 0.2417$

$x \sim N(\mu = 75, \sigma^2 = 10^2)$

$$P(80 < x < 90) = P\left(\frac{80-75}{10} < \frac{x-75}{10} < \frac{90-75}{10}\right)$$

$$= P(0.5 < z < 1.5)$$

$$= 0.9332 - 0.6915$$

b) Find score x → 95th percentile

$P(X \leq x) = 0.95$

$P\left(z \leq \frac{x-75}{10}\right) = 0.95$

$\frac{x-75}{10} = 1.645$

$x = 75 + 10 \times 1.645 = 91.45$

$P(z > 1.645) = 0.05$

$z_{0.5} = 0$

Linear Transformation of Normal Distribution

Thm: $x \sim N(\mu, \sigma^2)$ and $Y = aX + b$ then:

$$Y \sim N(a\mu + b, a^2\sigma^2)$$

Pf:

$$E[Y] = E[aX+b] = a\underbrace{E[X]}+b = a\underbrace{\mu}+b$$

$$Var(Y) = var(aX+b) = a^2 \underbrace{Var(X)} = a^2\underbrace{\sigma^2}$$

pretty much replacing definitions here

Normal Approximation to Binomial Distribution

For a binomial r.v $x \sim Bin(n, P)$ with:

$\mu = nP$

$\sigma^2 = nP(1-P)$

as $n \to \infty$, the distribution of x can be approximated by the normal distribution. $x \sim N(nP, nP(1-P))$

Pf

$$Z = \frac{X - E(X)}{\sqrt{var(X)}} = \frac{X - nP}{\sqrt{nP(1-P)}}$$

The mgf of Z converges to the mgf of the standard normal dist as $n \to \infty$

Conditions for Approximation

1)
- $nP > 5$
- $n(1-P) \geq 5$

2) Correction for Continuity:

Since the binomial dist is discrete and the normal dist is continous. Apply a continuity correction: $P(X \leq K)$: $P(X \leq K + 0.5)$

Example of Approximation

1) $x \sim Bin(n=25, \theta=0.4)$

Find $P(X=8)$

$$P(X=8) = \binom{25}{8} 0.4^8 = 0.120$$

2) $x \approx (nP = 25 \cdot 0.4 = 10, n\theta(1-\theta) = 6)$

Let $Y \sim N(10, 6)$

$$P(Y=8) = P(7.5 < Y \geq 9.5)$$

$$= \underbrace{P(-1.02 < z \; -0.61)}_{\text{StN Table}} = 0.117$$

1) Identify describing binomial distribution

2) check if normal approximation is appropriate $nP \geq 5$, $n(1-P) \geq 5$

3) Find $\mu = nP$, $\sigma = \sqrt{nP(1-P)}$

4) Apply normal distribution with continuity correction

5) Apply z score then find Probability

All Discrete Distributions

Formula:

Bernoulli → Models a single mutually exclusive independent trial → $\underbrace{p}_{\text{success}}{}^{x}\underbrace{(1-p)}_{\text{failures}}{}^{1-x}$ x is # of successes

Binomial → Follows n Bernoulli trials, counts the number of successes. → $\binom{n}{k} p^{k}(1-p)^{n-k}$

K successes in n trials

Bernoulli distribution is nothing but the Binomial distribution, where the trial only occurs once.

Binomial → $E(x) = n\underbrace{p}_{\text{wins}}$, $Var(x) = \underbrace{n\cdot p}_{\text{wins}}\underbrace{(1-p)}_{\text{Losses}}$

Geometric → Number of trials to first success → $\underbrace{(1-p)^{x-1}}_{\text{Failures}}\cdot\underbrace{p}_{\text{success}}$

Negative Binomial → Number of trials it takes to get certain # of Successes $\binom{k-1}{r-1}\cdot p^{r}(1-p)^{k-r}$

r is # of success
K is amt of trials

Hypergeometric → Drawing from finite population, without replacement $$\frac{\overbrace{\binom{M}{x}}^{\text{ways to choose}}\binom{N-M}{n-x}}{\binom{N}{n}}$$ ← ways to choose the "rest"

→ Total outcomes

Binomial questions have a set number of trials "out of n trials" "in n trials"

Negative Binomial questions have a set number of successes.

"number of successess"

Poisson Distribution → Models number of rare events that occur in fixed time interval

Poisson PMF: $\frac{\lambda^{k}\cdot e^{-\lambda}}{k!}$, $k=0,1,2$, $E[x]=\lambda$, $Var[x]=\lambda$

All Continous Distributions

Uniform Distribution → Models where every outcome in an interval is equally likely.

PDF: $f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{elsewhere} \end{cases}$

Note: α is the Start of the interval
β is the end of the interval

- Height of PDF is constant
- Area under curve is 1.

Exponential Distribution → Models the time in between Poisson processes

PDF:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

where λ = rate parameter
$f(x)$ decays as x increases

Probability of Intervals

$P(X \le t) = 1 - e^{-\lambda t}$
$P(X > t) = e^{-\lambda x}$
$P(X = x) = \lambda e^{-\lambda x}$

Gamma Distribution → The continous (waiting time) till an event can be described by an exponential distribution. For $r > 1$ events.

Actual Gamma Function: $\Gamma(\alpha) = \int_0^\infty t^{\alpha - 1} \cdot e^{-t} \, dt$

- Where α is the number of occurrences/events
- where β is the rate of the process.

PDF: Given

AS α increases, resembles normal distribution.

$\Gamma(\text{Positive int}) = (\text{Positive int} - 1)!$ → If α is not Pos int, use: $\Gamma(\alpha) = \int_0^\infty y^{\alpha - 1} \cdot e^{-y} \, dy$

To Solve Interval Questions

- $P(X \le 16) = F(16) = \int_0^{16} \frac{x^2 \cdot e^{-\frac{x}{8}}}{1024} \, dx$
- $P(8 \le x \le 32) = F(32) - F(8) = \int_0^{32} \frac{x^2 \cdot e^{-\frac{x}{8}}}{1024} \, dx - \int_0^{8} \frac{x^2 \cdot e^{-\frac{x}{8}}}{1024} \, dx$

All continous Distributions

Beta Distribution → Models random variables bound between 0 and 1. It is parameterized by α and β, both shape parameters.

PDF : $$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1$$

where $\Gamma(\alpha)$ is gamma function

The Distribution is Symmetric if $\alpha = \beta$

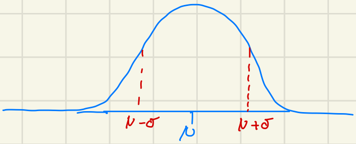
To Solve interval Questions

$f(X=0.5)$ – Plug in to PdF, use gamma function

$f(x \le 0.5)$ – ?

$f(a \le x \le b)$ – ?

Normal Distribution → This is Symmetric about it's mean, Shape follows a bell curve.



Key Characteristics

- Mean (μ): Determines centre of distribution
- Standard Deviation (σ): Determines how spread the curve is.

PDF: $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2(\sigma^2)}}$$

Mean: $E[X] = \mu$

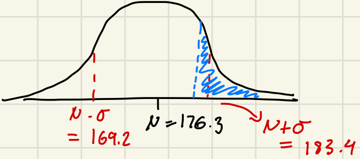
Variance : $V[X] = \sigma^2$

The Standard Normal Distribution

- IS when a normal distribution with $\mu = 0$ and $\sigma^2 = 1$, $\sigma = 1$

$$Z = \frac{X-\mu}{\sigma} \quad \text{Approx.} \quad \frac{x-np}{\sqrt{np(1-p)}}$$

EX



Normal CD : $P = 0.3$

To Find 90th Percentile, first Standardize: $z = \frac{x-\mu}{\sigma}$

Topic 5: Multivariate Distributions

Def: Multivariate Distributions Study the behaviour of multiple random variables at the same time.

Joint Distributions

- X and Y are two discrete r.v's
- The joint PMF is:

$$f(x,y) = P(X=x, Y=y)$$

This gives the probability $X=x$ and $Y=y$ at the same time. (the full picture)

Properties of Joint Dist

1.) $f(x,y) \geq 0$

2.) $\sum_x \sum_y f(x,y) = 1$

Adding up all probabilities for every possible pair of (x,y)

Note: If X and Y are indep the joint PMF will factor into $f(x,y) = f_X(x) \cdot f_Y(y)$

For Continous Joint Distributions:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

Marginal Distributions

- The marginal PMF describes the probability distribution of one of the r.V's on its own, ignoring the other var.

- To find marginal PMF of X, Sum joint probabilities $f(x,y)$ over all possible values of $\boxed{Y}$.

Discrete

$$f_X(x) = \sum_y f_{X,Y}(x,y)$$ → For x, sum across various values of y

$$f_Y(y) = \sum_x f_{X,Y}(x,y)$$ → For Y, sum across various values of x

Continous

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Conditional Distribution

The conditional probability $P(X=x \mid Y=y)$ is given by:

$$P(X=x \mid Y=y) = \frac{P(X=x,\ Y=y)}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)}$$

Divide joint pmf by Marginal pmf to get the conditional distribution

Joint PDF Overview

The joint pdf $f_{XY}(x,y)$ describes the probability density of 2 variables X and Y.

Joint PDF:

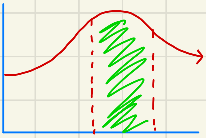
$$f_{XY}(x,y) \geq 0 \quad \text{for all } (x,y)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Example

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

The PDF is only valid from 0-1. Find c s.t it integrates to 1.



$$\text{Volume} = P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

Set up double integrals, for regions of interest

Def'n: Let X, Y be continous r.v.s with joint pdf $f(x,y)$. Then:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$ marginal pdf of x

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$ marginal pdf of Y

Continous Case

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

To Find Marginal PDF (continous)

1) Set bounds on integral
2) Set dx or dy, opposite to the marginal your solving for
3) Integrate

• Independence

- For discrete r.v, X,Y are indep, iff $f(x,y) = f_x(x) \cdot f_y(y)$ for all (x,y) in their range.
- For continous r.v, X,Y are indep iff $\underset{\text{Joint PDF}}{\underset{\downarrow}{f(x,y)}} = \underbrace{f_x(x) \cdot f_y(y)}_{\substack{\downarrow \\ \text{marginal PdF}}}$

$$f(x|y) = \frac{f(x,y)}{f_y(y)}$$

$$f(x,y) = f(x|y) \cdot f_y(y)$$

$$f_x(x) = \int_0^1 6xy \, dy = \frac{6x \cdot y^2}{2} \Big|_0^1 = 3x$$

$$f_y(y) = \int_0^1 6xy \, dx = \frac{6x^2 \cdot y}{2} \Big|_0^1 = 3y$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{3xy}{3x}$$

$$\int_0^1 \int_0^{1-x} f_{xy}(x,y) \, dy \, dx$$

$$\int_0^{\frac{1}{2}} \int_x^{\frac{1}{2}} 6(1-y) \, dy \, dx$$

$$\int_0^{\frac{1}{2}} \int_x^{\frac{1}{2}} 6-6y = 6y - \frac{6y^2}{2} \Big|_x^{\frac{1}{2}} = 3 - \frac{3}{4}$$

$$= \frac{9}{4} - 6x - 3x^2$$

How to Solve MGF Questions, Discrete Geometric Series

1) recognize discrete case
2) Simplify and get in correct form
3) Apply geometric series, by plugging in smallest value for k, then use this as numerator for geometric series
4) Plug back in and find valid bounds ($|r| < 1$)
5) Get Mgf and compute $E[x]$, $E[x^2]$, $V[x]$.

Practise Exam #1

1) 6 Books
10 Boxes

Each book has 10 options
So Total arrangements: 10^6

a) P(1 box has more than 1 book) = 1 − P(all books are in diff boxes)

$$P(\text{diff boxes}) = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5}{10^6}$$

Each book, has 1 less option than the last

P(diff boxes) = 0.1512

P(1 box has more than 1) = 1 − 0.1512
= 0.849

b) P(Box 1 or Box 2 or Both empty) = ?

First we must find the probability Box 1 is empty

$$P(\text{Box 1 empty}): \frac{9^6}{10^6} \qquad P(\text{Both empty}) = \frac{8^6}{10^6}$$
$$= 0.262$$

P(B1 empty) = 0.531

P(A∪B) = 0.531 + 0.531 − 0.262(2)

= 0.538

2)

urn 1	urn 2
- 5 whites	- 3 whites
- 3 greens	- 5 greens

a)

x:	0	1	2
$f(x)$:	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

To get 2 greens:

$$P(x=2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}}$$

$$P(x=2) = \frac{3}{28}$$

To get 1 green:

$$P(x=1) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}}$$

P(x=1) =

2b) Two mints from a are now in B, so it could be : 1W, 1G
2W, 0G
2G, 0W

urn 2: could be

4W	3W	5W
6G	7G	5G

$$P(BG|W) = \frac{P(W|BG) \cdot P(BG)}{P(W)}$$

Let BG represent both green

Let W represent white

urn 2
3W
5G + x

$$P(BG) = \frac{3}{28}$$

case of 2 greens

$$P(W|BG) = \frac{3}{10}$$

$$P(W) = P(W|x=0) \cdot P(x=0) + P(W|x=1) \cdot P(x=1) + P(W|x=2) \cdot P(x=2)$$
$$\frac{1}{2} \cdot \frac{10}{28} + \frac{4}{10} \cdot \frac{15}{28} + \frac{3}{28} \cdot \frac{3}{10}$$

$$P(W) = \frac{17}{40}$$

$$P(BG|W) = \frac{\left(\frac{3}{28} \cdot \frac{3}{10}\right)}{\frac{17}{40}}$$

$$P(BG|W) = \frac{9}{119}$$

3) S1 S2 S3 S4

a) 3 options:

1) 4 identical outcomes $P(4\text{ same}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16} \times \underbrace{2}_{\text{All heads or all tails}} = \frac{2}{16}$

2) 2 pairs of Heads/tails $P(2\text{ pairs}) = \frac{3}{8}$

3) Go again

Binomial (P=0.5, n=4)

$$\binom{n}{k} p^K (1-p)^{n-k}$$

$$\binom{4}{2}(0.5)^2(1-0.5)^2$$

a) To reach a desicion within one round means 1 minus the Probability that events A and Event B happened

$$1 - \frac{3}{8} - \frac{2}{16} = \frac{1}{2}$$

b) Let x reperesent the number of rounds

$P(x>3) = 1 - P(X=1) + P(X=2) + P(X=3)$

$P(x=1) = \frac{1}{2}$

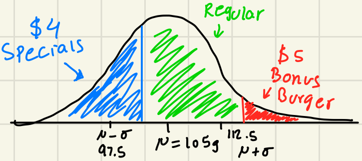
$P(x=2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$P(x=3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8}$

$= \frac{1}{8}$

4)



P= 0.744

We need to find the Probability that x lies within 125g and 100 g

We must Standardize:

$z = \frac{x-\mu}{\sigma}$

$z = \frac{125-105}{7.5}$

$z = \frac{8}{3} = 2.6$

$z = \frac{100-105}{7.5}$

$z = -0.66$

Now we find the Probability x lies between these bound, using table,

0.99609 $\quad$ 1 − 0.74537

0.25463

= 0.99609 − 0.25463

= 0.741

b) Need to find x value, which has 90% to its left.

Find corresponding z score to 0.9

$z = 1.28$

$z = \frac{x-\mu}{\sigma}$

$1.28 = \frac{x-105}{7.5}$

$7.5 \cdot 1.28 + 105 = x$

$x = 114.6$

5 b) $f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 < x \leq 1 \\ x - \frac{3}{4} & \text{if } 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$

a) ?

1) If $x < 0$

$\int_{-\infty}^{x} f(x)\, dx = 0$

2) $\int_{0}^{x} \frac{1}{4}\, dx = \frac{x}{4}\Big|_0^x = \frac{x}{4} - 0 + 0 = \frac{x}{4}$ @ $1 = \frac{1}{4}$

3) $\int_{1}^{x} x - \frac{3}{4}\, dx = \frac{x^2}{2} - \frac{3x}{4}\Big|_1^x = \frac{x^2}{2} - \frac{3x}{4} - \left[\frac{-1}{4} + \frac{1}{4}\right] = x^2 - \frac{3x}{4} + \frac{1}{2}$ @ 2

4) $\int_{2}^{x} = 1$

$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x}{4}, & 0 < x \leq 1 \\ \frac{x^2}{2} - \frac{3}{4}x + \frac{1}{2} & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$

c) $P(0.5 \leq x < 1.5) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

a) $\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{1} c\, dx + \int_{1}^{2} \left(x - \frac{3}{4}\right) dx = 1$

$= \int_{0}^{1} c\, dx + \quad \frac{x^2}{2} - \frac{3}{4}x\Big|_1^2 = \frac{1}{2} - -\frac{1}{4}$

$= \frac{1}{2} + \frac{1}{4}$

$= \frac{3}{4}$

$C = \frac{1}{4}$

6.) $f(x) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}, \; x = 0, 1, 2 \ldots$

a)

$M_{TX} = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{\lambda^x \cdot e^{-\lambda}}{x!}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$

$= \quad u = e^t \lambda$

$n! = x!$

$= e^{-\lambda} \cdot e^{e^t \lambda}$

$= e^{\lambda(e^t - 1)}$

b) The expected value is λ

$E[x] = \lambda$

$M'_X T = \lambda e^t \cdot e^{\lambda(e^t - 1)}$

7) $f(x) = \begin{cases} \frac{1}{2a} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

Show var of x is $\frac{a^2}{3}$

To find $Var(x)$, we need $E[x^2] - (E[x])^2$

$$E[x] = \int_{-a}^{a} x \cdot F(x)\, dx$$

$$E[x] = \int_{-a}^{a} x \cdot \frac{1}{2a}\, dx$$

$$E[x] = \frac{1}{2a}\int_{-a}^{a} x\, dx = \frac{1}{2a} \cdot \frac{x^2}{2}\Big|_{-a}^{a} = \frac{a^2}{2} - \frac{(-a)^2}{2}$$

$$E(x) = 0$$

$$E[x^2] = \int_{-a}^{a} x^2 \frac{1}{2a}\, dx$$

$$E(x^2) = \frac{1}{2a}\int_{-a}^{a} x^2\, dx = \frac{x^3}{3}\Big|_{-a}^{a} = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{2a^3}{3}$$

$$E[x^2] = \frac{a^2}{3}$$

$$V[x] = \frac{a^2}{3} - 0 = \frac{a^2}{3}$$

---

8) $f(x,y) = \begin{cases} 6(1-y) & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{, otherwise} \end{cases}$

Find $P(X \le \frac{1}{2}, Y \le \frac{1}{2})$

This means both x and y must be less than $\frac{1}{2}$.

Since $x \le y$, the region has bounds

- $0 \le x \le \frac{1}{2}$
  $x \le y \le \frac{1}{2}$

$$\int_{0}^{\frac{1}{2}}\int_{x}^{\frac{1}{2}} 6(1-y)\, dy\, dx$$

$$= 6\int_{x}^{\frac{1}{2}} (1-y)\, dy = 6\left(y - \frac{y^2}{2}\right)\Big|_{x}^{\frac{1}{2}} = \frac{9}{4} - \left[6x - 3x^2\right]$$

$$= \int_{0}^{\frac{1}{2}} \frac{9}{4} - 6x + 3x^2 = \frac{9}{4}x - \frac{6x^2}{2} + \frac{3x^3}{3}\Big|_{0}^{1/2} = \frac{1}{2}$$

Practise Exam #2

1) Fair die is tossed untill 6 occurs 4 times.
We are modelling the number of trials to get "r" successes.

Neg Binomial $\left(K=4,\ P=\frac{1}{6}\right)$

a) $P(X>6) = 1 - P(x\le 6)$

$P(X>6) = 1 - P(X=5) - P(X=4) - P(X=6)$

X can only take on velues 4,5 because
You need minimum 4 tosses to get 4 heads.

$P(X=5) = \binom{x-1}{K-1} p^K (1-p)^{x-K}$

$P(X=5) = \binom{4}{3}\left(\frac{1}{6}\right)^4\left(1-\frac{1}{6}\right)^1$

$P(X=6) = \binom{5}{3}\left(\frac{1}{6}\right)^4\left(1-\frac{1}{6}\right)^2$
$= 0.00535$

$P(X=5) = 0.002572$

$P(X=4) = \binom{3}{3}\left(\frac{1}{6}\right)^4\left(1-\frac{1}{6}\right)^0$
$= 0.00077$

$P(X>6) = 1 - 0.002572 - 0.0077 - 0.00535$

$P(X>6) = 0.991$

b) $P(X=10) = \binom{10-1}{4-1}\left(\frac{1}{6}\right)^4\left(1-\frac{1}{6}\right)^{10-4}$

Normally: $= 0.0217$

$P(X=10)$: within first 8 trials, You need 2 6's

Still models Neg Bin

$\binom{8}{2}\left(\frac{1}{6}\right)^2\left(1-\frac{1}{6}\right)^6$

$= 0.26 \cdot \underbrace{\left(\frac{1}{6}\right)^2}_{\text{Prob last 2 are heads}}$

$= 0.0072$

2) $P(\text{Hit with right}) = 0.7 \rightarrow P(\text{miss with right}) = 0.3$
$P(\text{Hit with left}) = 0.4 \rightarrow P(\text{miss with left}) = 0.6$

win: Two hits in a row

Let R repersent a hit with right hand
Let L reperesent hit with left hand
Let W repersent a win:

$$P(W \mid R) = \frac{P(R \mid W)\, P(W)}{P(R)}$$

P(W): The probability She wins can be,

$S = \{HHH,\ HHM,\ MHH,\}$

$P(W) = 0.7\cdot0.4\cdot0.7 + 0.7\cdot0.4\cdot0.3 + 0.3\cdot0.4\cdot0.7$

a) $= 0.364$

2b) P(W)= There are now 2 cases She wins
1 where She Starts with her right hand
So the chance of that one now becomes

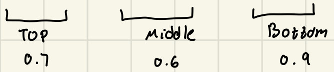
$P(W_1) = \frac{2}{3}\cdot(0.364)$
$P(W_1) = 0.2426$

This is the way She wins Starting with her left hand

$P(W_2) = 0.4\cdot0.7\cdot0.4 + 0.4\cdot0.7\cdot0.6 + 0.6\cdot0.7\cdot0.4$
$P(W_2) = \frac{1}{3}(0.448)$
$= 0.1493$

$P(W) = P(W_1) + P(W_2) = 0.391$

3)



Let D_1 reperesent finding corn in top drower

Let D_2 reperesent finding corn in mid drower

Let D_3 reperesent finding corn in bottom drower

$P(D_1) =$

$P(D_2) = \frac{1}{3} \rightarrow P(D_1' | D_2) = 1$, $P(D_1') = P(D_1' | D_1) \cdot P(D_1) + P(D_1' | D_2) \cdot P(D_2) + P(D_1' | D_3) \cdot P(D_3)$

$P(D_3) =$

$P(D_1') = 0.3 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}$

$P(D_2 | D_1') = \frac{P(D_1' | D_2) \cdot P(D_2)}{P(D_1')}$ $P(D_1') = \frac{23}{30}$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{23}{30}}$$

$$= 0.435$$

$P(D_2) = P(D_1') \cdot P(D_2')$

$= 0.3 \cdot 0.1$

$= 0.3$

7) $f(x) = \begin{cases} K x^6 (1-x)^9 & , \text{if } 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$\beta - 1 = 9$

$\alpha - 1 = 6$

$\therefore \beta = 10$

$\alpha = 7$

a) To find K: Apply constant from beta distribution, $\frac{\Gamma(7+10)}{\Gamma(7)\Gamma(B)}$ $\therefore \frac{16!}{6! \cdot 9!} = 80080$

$K = 80080$

b) To find $E[x^3]$: We get the mean,

$$\int_0^1 x^3 \cdot 80080\, x^6 (1-x)^9$$

$$\int_0^1 80080\, x^9 (1-x)^9$$

$$= 80080 \int_0^1 \underbrace{x^9 (1-x)^9}_{\text{Describes beta function}} dx$$

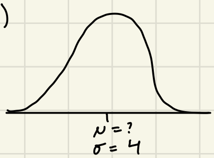
$\beta = 10$ Plug in to Constant

$\alpha = 8$ formula: $\frac{1}{24310}$

$= 80080 \cdot \frac{1}{24310} = 3.29$

$E[x^3] = 3.29$?

4)



$P(x < 32) = 0.02$

$z = \frac{x - \mu}{\sigma}$

The corresponding z Score to 0.02 is

$1 - 0.98$

$z = -2.6$

6.508.

$-2.6 = \frac{32 - \mu}{4}$

$4(-2.6) + 32 = 42.4$

b) $P(x > 48)$

$P(x > 48) = P\left(z > \frac{48 - \mu}{\sigma}\right)$

$= P(z > 1.95)$

$P(z > 1.95) = 1 - 0.97441$

$= 0.0256$

$P(1 \text{ in } 4 \text{ years}) = 1 - P(\text{none of years} > 48)$

$= 1 - 0.0256$

$= 0.9744$

5) $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 1.5 \\ 0 & \text{elsewhere} \end{cases}$

a)

1) For $x \le 0$

$\int_{-\infty}^{x} f(x)\,dx = 0$

2) For $0 \le x \le 1$

$\int_0^x x\,dx = \frac{x^2}{2}\Big|_0^x = \frac{x^2}{2} - 0 + 0 = \frac{x^2}{2}$ @ 1 $= \frac{1}{2}$

3) For $1 \le x$ 1.5

$\int_1^x 1\,dx = \quad x\Big|_1^2 = x - 1 + \frac{1}{2} = x - \frac{1}{2}$

4) For $x \geq 1.5$

$\int_{1.5}^{x} f(x)\,dx = 0$

b) $Y = 400x - 50$

Find σ: To find this, I would find the variance using $V(x) = E[x^2] - (E[x])^2$

$V(Y) = E[Y^2] - (E[Y])^2$

$V(ax+b) = aE[Y] + b$

$=$

$E[Y] = \int_0^1 x \cdot (400x - 50) + \int_1^{1.5} x \cdot (400x - 50)$

$\int_0^1 400x^2 - 50x \quad + \int_1^{1.5} 400x^2 - 50x$

$= \frac{400x^3}{3} - \frac{50x^2}{2}\Big|_0^1 + \frac{400x^3}{3} - \frac{50x^2}{2}\Big|_1^{1.5}$

$= 108.3 + 285.416$

$E[Y] = 393.716$

$= 400(393.716) - 50$

$V = 157436.4$

$\sigma = \sqrt{157436.4}$

$\sigma = 396.782$

b) $P(X < 2 \mid Y < 1) = \frac{P(X < 2, Y < 1)}{P(Y < 1)}$

Marginal pdf of Y: $\int_y^\infty e^{-x}\,dx = \frac{e^{-x}}{-1}\Big|_y^\infty = 0 + e^{-y}$

$f_Y(y) = e^{-y}$

Now plug in Y: $\int_0^1 e^{-y} = \frac{e^{-y}}{-1}\Big|_0^1 = 1 - e^{-1}$

$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 \le x \le 1 \\ x - \frac{1}{2} & 1 < x \le 1.5 \\ 1 & \text{if } x > 1.5 \end{cases}$

$F(1.2) - F(0.5)$

$= \frac{7}{10} - \frac{1}{8}$

$= 0.875$

$\frac{3}{2} - \frac{1}{2}$

6) Geometric: $f(x) = p(1-p)^{x-1}$

This is a discrete distribution.

$M_{TX} = \sum_{K=0}^{\infty} e^{tx} \cdot f(x)$

$M_{TX} = \sum_{K=0}^{\infty} e^{tx} \cdot p(1-p)^{x-1}$

$= p \sum_{K=0}^{\infty} e^{tx} \cdot (1-p)^{x-1}$

$= p \sum_{x=0}^{\infty} [e^t(1-p)]^{x-1} \cdot e^t$

$= pe^t \sum_{x=0}^{\infty} \frac{1}{1 - e^t(1-p)} \quad r = e^t(1-p)$

$M_X(t) = \frac{pe^t}{1 - (1-p)e^t}$

b)

Set derivative equal to 0.

$\frac{dp}{dt} = \frac{pe^t}{1 - (e^t - pe^t)}$

8.) $f(x,y) = \begin{cases} e^{-x}, & \text{if } 0 \le y \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$

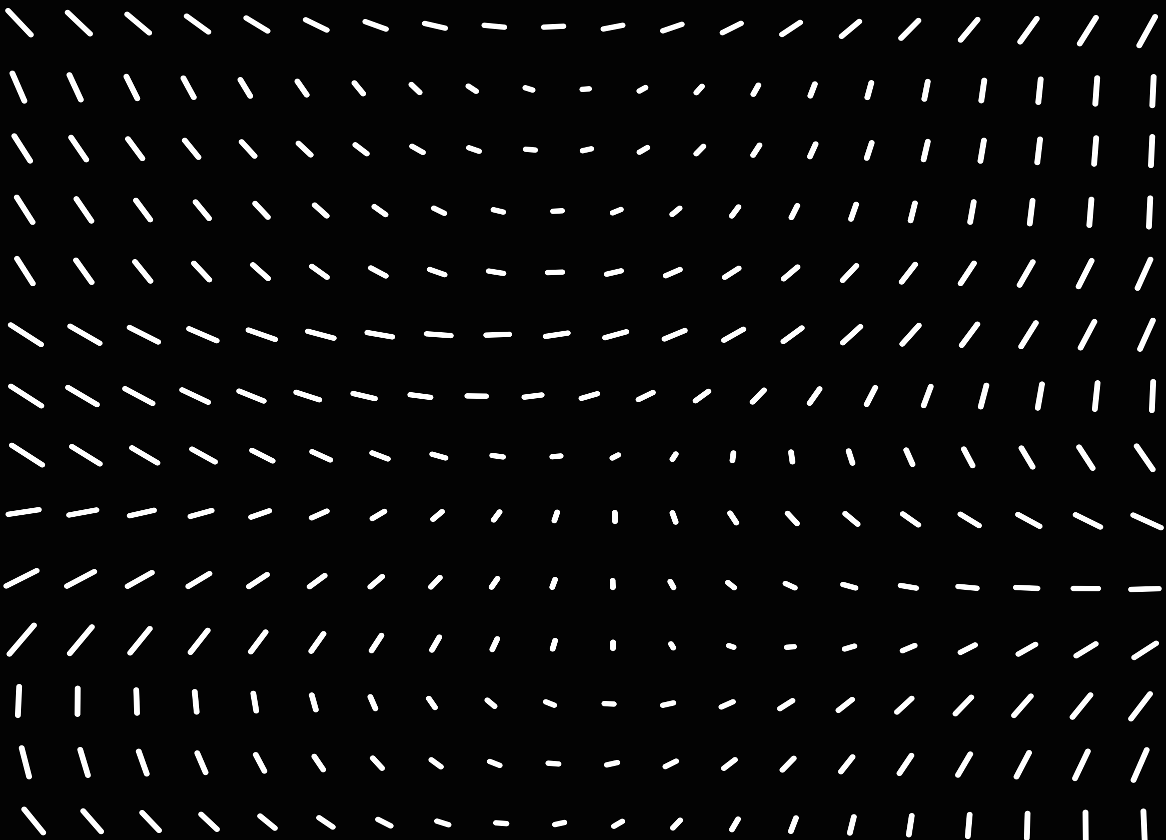
a) $P(X \ge 2Y) = 1 - P(X < 2Y)$

$\int_0^\infty \int_{2y}^\infty e^{-x}\,dx\,dy = -e^{-x}\Big|_{2Y}^\infty = e^{-2Y}$

$\int_0^\infty e^{-2Y}\,dy = \frac{e^{-2Y}}{-2}\Big|_0^\infty = 0 - -\frac{1}{2}$

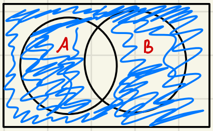
$= \frac{1}{2}$

Exam Practise



Assignment 1- Practise

1) a) $(A \cap B)' = A' \cup B'$



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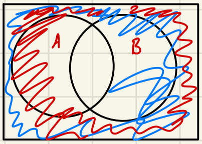


2) 52c5 - Total cards

a) $P(2\text{ pairs}) = \frac{\binom{13}{1}\cdot\binom{4}{2}\ \binom{12}{1}\binom{4}{2} + \binom{11}{1}\binom{4}{1}}{52c5}$?

$= \frac{247104}{2598960} \longrightarrow 0.095$

Note: If its a 5 card hand, dont take one rank, then the other (overcounting).

b) $\frac{\binom{13}{1}\cdot\binom{4}{4}\ \binom{12}{1}\cdot\binom{4}{1}}{52c5}$

$= 0.000240$

3) Choosing 4 of 6 cities.

$C = \{T, M, O, W, E, V\}$

a) $6P4 = 360$ ways

b) $4P2 = 12$ ways $\times\ 4P2$

$= 144$ ways

c) $\frac{\frac{P^4_2}{2} \cdot P^4_2}{P^4_2 \cdot P^4_2} = \frac{1}{2}$

4) $L = \{A \ldots\ldots Z\} = 26$ options

$N = \{0, 1, 2 \ldots 9\} = 10$ outcomes

Total ways to make a valid License plate: $26^3 \cdot 10^3$

a) All 3 letters same means $\binom{26}{1}$ choosing one letter all 3 times.

$P(\text{same letter}) = \frac{26 \cdot 10^3}{26^3 \cdot 10^3} = 0.0015$

b) $P(\text{all even or all odd}) = \frac{26^3 \cdot 5^3}{26^3 \cdot 10^3} + \frac{26^3 \cdot 5^3}{26^3 \cdot 10^3}$

$= \frac{1}{4}$

*

c) $\frac{26^3 \cdot 9^3}{26^3 \cdot 10^3} = 0.729$

$= 1 - 0.729$

$= 0.271$

6) F1
F2
F3
F4
F5

C1
C2
C3

a) To find sample points we can assign any F1 - to any 3. So permutation can be used because order matters $5P3 = 60$

b) $\binom{4}{2} \cdot 3! \to \frac{36}{60} = 0.6$

7) Let O reperesent off campus
Let V reperesent virginia

$P(O) = \frac{1}{3}$ $P(O') = \frac{2}{3}$

$P(V) = \frac{5}{9}$ $P(V') = \frac{4}{9}$

$P(V' \cup O') = \frac{3}{4}$

$P(V' \cap O') = ?$

$P(V' \cup O') = P(V \cap O)'$

$P(V \cap O)' = \frac{3}{4}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(V' \cup O') = P(O') + P(V') - P(O' \cap V')$

$\frac{3}{4} = \frac{2}{3} + \frac{4}{9} - P(O' \cap V')$

$P(O' \cap V') = \left(\frac{2}{3} + \frac{4}{9}\right) - \frac{3}{4}$

$P(O' \cap V') = \frac{13}{36}$

8) 100 tickets
5 winners
96 tickets after organizers buy one each

$P(\text{no prize}) = 1 - P(\text{at least 1 wins prize})$

$\binom{100}{5}$ ways to choose a winner.

If at least 1 wins: $\frac{\binom{4}{1}\cdot\binom{96}{4}}{\binom{100}{5}} = 0.1765$

$1 - 0.1765 = \underline{0.8235}$

82.35% chance to not win prize.

or $\frac{\binom{96}{5}}{100C5}$

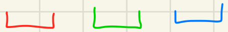
Assignment 2

1.) If $P(B) \neq 0$, then

$$P(A|B) \geq 0 \rightarrow \frac{P(A \cap B)}{P(B)} \geq 0$$

$$P(B|B) = 1 \rightarrow P(B|B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

3) $\boxed{1}$ $\boxed{2}$



Total ways to assign 2 books to 3 boxes is Each book has 3 options:

3x3 ways to arrange

a) $$S = \left\{ \begin{array}{ccc} B11 & B22 & B33 \\ 12 & 12 & 12 \\ B13 & B12 & B23 \\ 12 & 12 & 12 \\ B31 & B32 & B21 \\ 12 & 12 & 12 \end{array} \right\}$$

$P(A) = \frac{7}{9}$ $P(C) = \frac{3}{9}$

$P(B) = \frac{3}{9}$ $P(B \cap C) = \frac{1}{3} \cdot \frac{1}{3}$

$\frac{1}{9} = \frac{1}{9}$ ✓

B and C are indep

They are not mutually exclusive

4) $\boxed{W}$ $\boxed{B}$

$A = \{$1st dice : 1, 2, 3$\} \longrightarrow P(A) = \frac{1}{2}$

$B = \{$1st dice: 3, 4, 5$\} \rightarrow P(B) = \frac{1}{2}$

$C = \{$sum is 9$\} \longrightarrow P(C) = ?$ $\frac{1}{9}$

36 ways: { 11, 12, 13, 14, 15, 16

$\frac{4}{36}$

21, 22, 23, 24, 25, 26

31, 32, 33, 34, 35, $\underline{36} \rightarrow 2$

41, 42, 43, 44, $\underline{45}$, 46 $\boxed{36,}$ 63, 45, 54

$P(A \cap B \cap C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{36}$ ✓ Correct

They are not indep

as

$P(A \cap B) \neq P(A) \cdot P(B)$

$P(B \cap C) \neq P(B) \cdot P(C)$

$P(A \cap C) \neq P(A) P(C)$

5) Let A reperesent smn seeing the add

Let P reperesent smn purchasing

Let T reperesent TV

Let M reperesent magazine

$P(M) = \frac{1}{50}$ $P(P) = ?$

$P(P) = P(P|A) \cdot P(A) + P(P|A') \cdot P(A')$

$P(T) = \frac{1}{5}$

$P(M \cap T) = \frac{1}{100}$

$P(P|A) = \frac{1}{3}$

$P(P|A') = \frac{1}{10}$

$P(A) = P(M \cup T)$

$$P(M \cup T) = P(M) + P(T) - P(M \cap T)$$
$$= \frac{1}{50} + \frac{1}{5} - \frac{1}{100}$$

$P(M \cup T) = \frac{21}{100}$

$P(A) = \frac{21}{100}$ $P(P) = \frac{1}{3} \cdot \frac{21}{100} + \frac{1}{10} \cdot \frac{79}{100}$

$P(A') = \frac{79}{100}$ $= 0.149$

6) 100C2 - Total ways

a) Hyper geometric:

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \frac{\binom{25}{1}\binom{40}{1}}{\binom{100}{2}}$$

$$P(1r, 1w) = \frac{20}{99}$$

b) $$= \frac{25}{100} \cdot \frac{40}{100} + \frac{40}{100} \cdot \frac{25}{100} = 0.2$$

Assignment 3

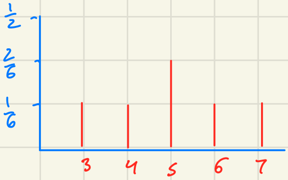
1) ① ② ③ ④ $S = \{12, 13, 14, 23, 24, 34\}$

2 Balls drawn from 4

Z reperesents the sum of 2 balls

a) The possible values X can take:

$f(x)$:	$x=3$	$x=4$	$x=5$	$x=6$	$x=7$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$



b)
$$F(x) = \begin{cases} 0 & \text{if } x < 3 \\ \frac{1}{6} & \text{if } 3 < x < 4 \\ \frac{2}{6} & \text{if } 4 \leq x < 5 \\ \frac{4}{6} & \text{if } 5 \leq x < 6 \\ \frac{5}{6} & \text{if } 6 \leq x < 7 \\ 1 & \text{if } 7 \leq x < 8 \end{cases}$$

2) $P(X \leq 3) = \frac{3}{4}$

a)

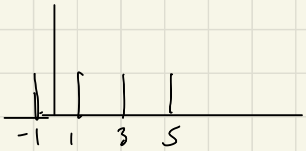
$P(X < 3) = \frac{1}{2}$

$P(X \geq 1) = 1 - P(X < 1) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(-0.4 < x < 4) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

$P(x = 5) = 1$

b)
$$f(x) = \begin{cases} \frac{1}{4}x & \text{for } -1 \leq x < 1 \\ \frac{1}{4}x & \text{for } 1 \leq x < 3 \\ \frac{1}{4}x & \text{for } 3 \leq x < 5 \\ \frac{1}{4}x & \text{for } x \geq 5 \end{cases}$$



3)

	Frequency	
0	1	$\frac{1}{40}$
1	6	$5/40$
2	16	$16/40$
3	13	$13/40$
4	4	$4/40$
5	1	$\frac{1}{40}$

$P(X = x)$ Odd= $\frac{19}{40}$

$$\frac{\binom{19}{x}\binom{18}{5-x}}{\binom{37}{5}}$$

For $x = 0, 1, 2, 3, 4, 5, \cancel{6}$

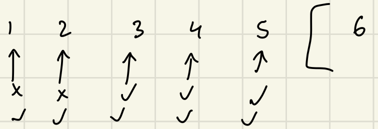
Models it well

1) $P(O+) = \frac{2}{5}$

$P(O-) = \frac{1}{15}$

a) More than 5 ppl untill 3 successes
To get r successes this models neg bin

Neg Bin $\sim (k = 3,\ P = \frac{2}{5},\ x = 5)$



$P(\text{More than 5 ppl}) = 1 - P(X=3) + P(X=4) + P(X=5)$

$P(X=3) = \binom{3-1}{3-1}\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^{3-3}$

$= 0.064$

$P(X=4) = \binom{4-1}{3-1}\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^{4-3}$

$= 0.1152$

$P(X=5) = \binom{5-1}{3-1}\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^{2}$

$= 0.13824$

$P(X>5) = 1 - 0.13824 - 0.1152 - 0.064$

$= 0.6825$

b) $P(x \geq 10) = 1 - [P(X=9) + P(X=8)]$

$P = \frac{1}{15}$

$P(X=9) = \frac{1}{15}\left(1-\frac{1}{15}\right)^8$

$= \frac{1}{15}\left(\frac{14}{15}\right)^8 = \dfrac{\frac{1}{15}}{1-\left(\frac{14}{15}\right)}$

Assignment 4

1) Show variance: σ^2

$$V(ax+b) = a^2\sigma^2$$

2) 8 Pairs of Shoes

16 total Shoes

$\binom{16}{4}$ - ways for her to randomly takeout 4 shoes.

	0	1	2
$f(x)$	$\frac{8}{13}$	$\frac{24}{65}$	$\frac{1}{65}$

$$y = 4x - 1(4 - \underset{\sim}{2x})$$

$$y = 4x - 4 + 2x$$

$$y = 6x - 4$$

must be wrong in pairs

$$E[X] = 0.4$$

$$E[ax+b] = aE[x] + b$$

$$= 6[0.4] - 4$$

$$E[Y] = \frac{-8}{5}$$

3a) $$\int_3^9 \frac{1}{2} e^{-\frac{(x-3)}{2}} + 0$$

$$\frac{1}{2}\int_3^9 e^{-\frac{(x-3)}{2}} dx$$

$$= \frac{1}{2}\cdot \frac{e^{-\frac{(x-3)}{2}}}{-\frac{1}{2}}\Bigg|_3^9 = -0.04978 - -1$$

$$= 0.95022$$

$$-\frac{(x-3)}{2}$$

$$\frac{(-x+3)}{2} \qquad \frac{-1}{2}$$

3b) $N = 5$ $\cdot V = 4$

$\sigma = 2$

$$K = \frac{9-5}{2} = 2$$

$K = 2$

Lower Bound: $1 - \frac{1}{K^2} = 1 - \frac{1}{4}$

$= 75\%$

eh kinda?

4) $$F(x) = 2\left(\frac{1}{3}\right)^x \text{ for } x = 1, 2, 3....$$

This is a discrete distribution:

$$\sum_{K=0}^{\infty} ar^K = \frac{a}{1-r} \text{ for } |r|<1$$

$$M_{TX} = \sum_{K=0}^{\infty} 2\left(\frac{1}{3}\right)^x \cdot e^{tx}$$

$$M_{TX} = 2\sum_{K=0}^{\infty}\left(\frac{e^t}{3}\right)^x$$

Apply geometric Series: $r = \frac{e^t}{3}$, $a = 1$

$$\frac{e^t}{3} < 1$$

$$e^t < 3$$

$$\cancel{\ln}\,\cancel{e}^t < \ln 3$$

$$t < \ln 3$$

$$= 2\cdot\frac{1}{1 - \frac{e^t}{3}} = \frac{2}{\left(1 - \frac{e^t}{3}\right)}$$

$$= \frac{2\left(\frac{e^t}{3}\right)}{1 - \frac{e^t}{3}} \text{ for } t < \ln 3$$

$$= \frac{2e^t}{3 - e^t}$$

$$f' = \frac{2e^t(3-e^t) - (-e^t)(2e^t)}{(3-e^t)^2}$$

$$f' = \frac{6e^t - \cancel{2e^{t^2}} + e^t + \cancel{2e^{t^2}}}{(3-e^t)^2}$$

$$f' = \frac{6e^t}{(3-e^t)^2}\Bigg|_0$$

$$f'' = \frac{(6e^t)(3-e^t)^2 - (-e^t)^2(6e^t)}{(3-e^t)^4}$$

Comeback to

5a) $$M_{TX} = \int_{-\infty}^{\infty} e^{tx}\cdot\frac{1}{2}e^{-|x|}$$

$$= \int_0^{\infty} e^{tx}\cdot\frac{1}{2}e^{-x} dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{(1+t)x} dx + \frac{1}{2}\int_0^{\infty} e^{(t-1)x} dx$$

$$= \frac{1}{2}\frac{e^{(t+1)x}}{1+t}\Bigg|_{x=-\infty}^{0} + \frac{1}{2}\frac{e^{(t-1)x}}{(t-1)}\Bigg|_{x=0}^{\infty}$$

$$= \frac{1}{2(t+1)}$$

7) $\mu = 0.260$

$\sigma = 0.005$

$$1 - \frac{1}{K^2} = \frac{35}{36}$$

$$\frac{1}{36} = \frac{1}{K^2}$$

$$K = 6$$

This means mean μ must fall within $K\sigma$ or $6 \cdot 0.005$ ± the mean as the interval

$$\mu + 0.03$$

$$\left.\begin{array}{l} 0.260 + 0.03 = 0.29 \\ 0.260 - 0.03 = 0.23 \end{array}\right\} \text{Intenal}$$

8) 8 questions

3 answers

$\frac{1}{3}$ Chance going for any of the answer.

$P(X=4)$: $8P3 = 336$ ways to answer for the test

$$\binom{n}{K} p^K (1-p)^{n-K} = \binom{8}{4} \left(\frac{1}{3}\right)^4 \left(1 - \frac{1}{3}\right)^4$$

$$P(X=4) = 0.1365$$

$$P(X \geq 2) = 1 - P(X=0) + P(X=1)$$

$$P(X \geq 2) = \binom{8}{0} \left(\frac{1}{3}\right)^0 \left(1 - \frac{1}{3}\right)^8 + \binom{8}{1} \left(\frac{1}{3}\right)^1 \left(1 - \frac{1}{3}\right)^7$$

$$= 0.195$$

$$1 - 0.195 = 0.805$$

Assignment 5

1) Busy 60% of the time

a) $P(x=3)$ Probability of success is 0.4

$P(x=3) = (0.4)^2(0.6)$

geometric : $0.4(1-0.4)^{3-1}$

$P(x=3) = 0.144$

$P(X \leq 3) = P(x=1) + P(x=2) + P(x=3)$

$P(x \leq 3) = 0.4 + 0.6 \cdot 0.4 + 0.6^2 \cdot 0.4$

$= 0.784$

b) If both call independently, we still have a 0.4 chance of getting through.

$P(X \geq 4) = 1 - P(x=2) + P(x=3) + P(x=4)$

X can not take on 1, because theres no way both of us can get through in one call

$P(X=2)$: we both get first try : $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B)$: $0.4 \cdot 0.4$

$P(X=2) = 0.16$

$P(x=3)$: one of us firs try, other second try:

Neg Binomial

$P(x=3) \binom{3-1}{2-1}(0.6)^{3-2} \cdot 0.4^2$

$P(X \geq 4) =$

$= 1 - 0.16 - 0.192$

$P(X \geq 4) = 0.648$

2) 2 Male
3 Females

For males : $P(R) = 0.2$

For females: $P(R) = 0.4$

2 patients selected without replacement.

a) Probability 2 females:

Hypergeometric:

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$P(2F) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}}$$

$P(2F) = 0.3$

a) $P(\text{Diff genders}) : \dfrac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}$

$P(\text{Diff genders}) = 0.6$

b) Probability, they will respond is like expected value:

we have 3 cases

1M, 1F – Both respond → $0.6 \cdot 0.2 \cdot 0.4$

2M, 0F – Both respond → $0.1 \cdot 0.2^2$

0M, 2F – Both respond → $0.3 \cdot 0.4^2$

Add All these

$P(R) = 0.1$

3) PMF : $\dfrac{\lambda^K \cdot e^{-\lambda}}{K!}$ $\displaystyle\sum_{K=0}^{\infty} e^{tx} \cdot f(x)\,dx$

Need to prove $V(X) = \lambda$

$E[X(X-1)]$ By finding mgf:

$$= \sum_{K=0}^{\infty} \frac{\lambda^K \cdot e^{-\lambda}}{K!} \cdot e^{tx} = e^{-\lambda + tx}$$

4) $\lambda = 5.2$
$f(3, 5.2) = 0.1293$

5) $P = 0.0012$ Poisson: $\dfrac{\lambda^K \cdot e^{-\lambda}}{K!}$, where
$\lambda = 0.0012$
$K = 0, 1, 2$?
Maybe jus 2?

Poisson approximation to binomial

Binomial is independent, fixed trials.

Poisson is rare independent events.

Binomial $(P = 0.0012, n = 1000)$ $\lambda = nP$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{n}{k}(p)^k(1-p)^{n-k}$ $p = 0.87959$
$p = 1 - 0.87959$
$P = 0.12041$

$$P(X \leq 2) = \frac{0.0012^2 \cdot e^{-0.0012}}{2}$$

$P(X \leq 2) = 7.19 \times 10^{-7}$

λ is just equal to np

Just Solve with $\lambda = 1.2$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= 0.879$

8) Mean: β

$X = K$ corresponds to $K-1 \leq Y < K$.

$P(X=K) = P(K-1 \leq Y < K) = \displaystyle\int_{K-1}^{K} f_Y(y)\,dy$

$P(X=K) = \displaystyle\int_{K-1}^{K} \frac{1}{\beta} e^{-y/\beta}$

$P(X=K) = -e^{-K/\beta} + e^{-(K-1)/\beta}$

Recognize geometric Series

$P(X=K) = (1-\theta)\theta^{K-1}$, $K = 1, 2$

Assignment 6

1) Given gamma distribution

Show $M_{TX} = \frac{1}{(1-\beta t)^{\alpha}}$

$$M_{TX} = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \, dx$$

$$M_{TX} = \int_{-\infty}^{\infty} \frac{x^{\alpha-1} (e)^{tx - \frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)}$$

$$M_{TX} = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_{-\infty}^{\infty} x^{\alpha-1} (e)^{-x\left(\frac{1}{\beta}-t\right)}$$

To ensure it goes to 0, we must $-x\left(\frac{1}{\beta}-t\right) < 0$

$t < \frac{1}{\beta}$

$$= \int_{0}^{\infty} e^{-y} \frac{y^{\alpha-1}}{\left(\frac{1}{\beta}-t\right)^{\alpha-1}} \frac{dy}{\left(\frac{1}{\beta}-t\right)}$$

Let $y = x\left(\frac{1}{\beta}-t\right)$

$x = \frac{y}{\left(\frac{1}{\beta}-t\right)}$

$dx = \frac{1}{\left(\frac{1}{\beta}-t\right)} dy$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)\left(\frac{1-\beta t}{\beta}\right)^{\alpha}} \int_{0}^{\infty} y^{\alpha-1} \cdot e^{-y} dy$$

$$= (1-\beta t)^{-\alpha}, \text{ when } t < \frac{1}{\beta}$$

b) $m(t) = \frac{1}{(1-\beta t)^{\alpha}}$, when $t < \frac{1}{\beta}$

$$m'(t) = 1\,\alpha \cdot (1-\beta t)^{\alpha-1} (-\beta)$$

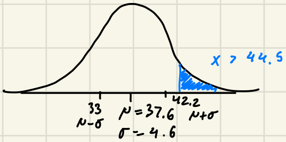
$$= \frac{-\beta\alpha}{(1-\beta t)^{\alpha-1}} \longrightarrow E[x] = m'(0) = \alpha\beta$$

$$m''(t) = -\beta\alpha(\alpha-1)(1-\beta t)^{(\alpha-1)-1} \longrightarrow E[x^2] = \alpha(\alpha+1)\beta^2$$

$$V(X) = E[x^2] - (E[x])^2$$

$$V(X) = \alpha\beta^2$$

2.)



$P(x \geq 44.5) = 0.066$

$P(x \geq 44.5)$: $z = \frac{44.5 - 37.6}{4.6}$

$z = 1.5$

z of 1.5 corresponds to 0.93319

a) $P(x \geq 44.5) = 1 - 0.93319$
$= 0.00681$

c) $P(30 \leq x \leq 40) = 0.6498$

b) $P(x \leq 35) =$

$z = \frac{x - \mu}{\sigma}$

$z = -0.565$

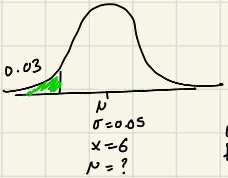
$z = 0.71226$

$z = 1 - 0.71226$

0.28774

$P(x \leq 35) = 0.28774$

3)



$0.03 = 1 - 0.97$

On the z table this corresponds to

-1.88 or 1.88

$$z = \frac{x-\mu}{\sigma}$$

$$1.88 = \frac{6-\mu}{0.05}, \quad \mu = 6.094$$

4) $p = 0.23$, $n = 120$, $x = 32$

Normal approximation

1) Does it make sense?

$np \geq 5$? $\quad np(1-p) \geq 5$?

$\mu = 27.6$ ✓

$\sigma^2 = 21.252$

$\sigma = \sqrt{21.252}$

2) $P(X > 32) = 1 - P(X \leq 31)$

$P(X > 32) = P(X \geq 32.5)$ Correction for continuity

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{32.5 - 27.6}{\sqrt{21.252}}$$

$z = 1.063$

$P(Z > 1.06) = 1 - P(Z \leq 1.06)$
$= 1 - 0.85543 = 0.1446$

5) $_{52}C_2$ - Total ways to draw 2 cards

Let Z represent the number of aces

Let W represent total aces

a) To find the joint pmf of Z and W. We will set up table, with possible values.

W \ Z	0	1
0	$\frac{188}{221}$	0
1	$\frac{16}{221}$	$\frac{16}{221}$
2	0	$\frac{1}{221}$

$\frac{48}{52}$

Marginal PMF of Z

$\frac{188}{221} + \frac{16}{221} = f(0)$

$f(z=0) = \frac{204}{221}$

$f(z=1) = \frac{17}{221}$

c) $P(W \mid Z=1) = \frac{P(W, Z=1)}{f_Z(z)}$

$= \frac{\frac{16}{221}}{\left(\frac{17}{221}\right)} = \frac{16}{17}$ for $w = 1$

$= \frac{1/221}{\frac{17}{221}} = \frac{1}{17}$

6.) $f(p,s) = \begin{cases} 5pe^{-5s} & \text{for } 0.2 < p < 0.4,\ s > 0 \\ 0 & \text{elsewhere} \end{cases}$

a) Bounds

$s > 2$

$p < 0.3$ → but greater than 0.2

$$\int_{0.2}^{0.3}\int_{2}^{\infty} 5pe^{-5s}\, ds\, dp$$

$$= \int_{2}^{\infty} 5pe^{-5s}\, ds\, dp = 5p\int_{2}^{\infty} e^{-5s}\, ds = 5p\frac{e^{-5s}}{-5}\Big|_{2}^{\infty} = 5e^{-2p}$$

$$= \int_{0.2}^{0.3} 5e^{-2p}\, dp = 5\int_{0.2}^{0.3} e^{-2p} = 5 \cdot \frac{e^{-2p}}{-2}\Big|_{0.2}^{0.3}$$

$= -1.372 - -1.675$

$= 0.304$

b) $$\int_{0.25}^{0.3}\int_{0}^{1} 5pe^{-ps}\, ds\, dp$$

$$5p\int_{0}^{1} e^{-ps} = \frac{e^{-ps}}{-p}\Big|_{s=0}^{s=1} = \frac{e^{-p}}{-p} - \frac{1}{-p} = \frac{e^{-p}-1}{-p}$$

$$\int_{0.25}^{0.3} \frac{e^{-p}-1}{-p}\, dp = 0.06$$

7) x is amt of dollars spent on gasoline.
y is amt of dollars reimbursed

$$f_x(x) = \int_{-\infty}^{\infty} f(x)\, dy\, dx$$

$$\int_{10}^{20}\int_{\frac{x}{2}}^{x} \frac{1}{25}\left(\frac{20-x}{x}\right)$$

$$= \frac{1}{25}\int_{\frac{x}{2}}^{x} (20-x)(x^{-1})$$

$$20x^{-1} - x \longrightarrow \frac{1}{25}\left(20x - \frac{x^2}{2}\right)\Big|_{\frac{x}{2}}^{x}$$

$$= \frac{20}{25}x - \frac{x^2}{50} - \left[\frac{10x}{25} - x^2\right]$$

$$\int_{10}^{20} \quad = f_x(x) = \frac{20-x}{50}$$

$$f_x(x) = \int_0^1\int_0^1 6xy\, dy\, dx$$

$$= 6\int_0^1 xy\, dy = 6 \cdot \frac{xy^2}{2}\Big|_0^1 = 3x \rightarrow F_x(x) = 3x$$

$$= \int_0^1 6xy\, dx$$

$$= 6\int_0^1 xy\, dx = 6 \cdot \frac{x^2}{2}y\Big|_0^1 = 3y$$

$$f(y|x) = \frac{f_{xy}(x,y)}{f_x(x)}$$

→ original function

→ probability of x

$$= \frac{6xy}{3\checkmark}$$