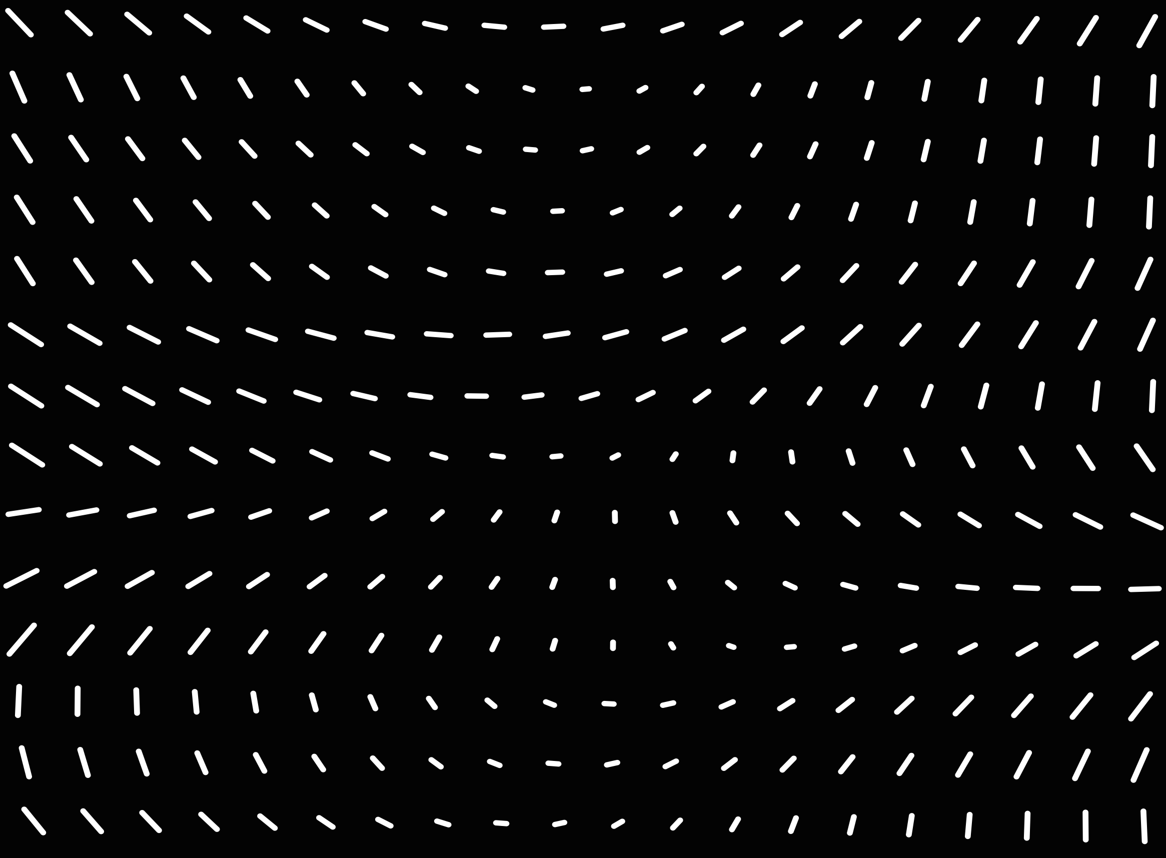
Math 221- Vector Calculus

All Lecture Notes



Summarized Notes

Lecture 2 Notes

Linear Functions

General form of a line $f(x) = mx + b$

- where m is the slope, b is the y intercept.

Slopes

- The slope, is the ratio of rise over run for a linear function.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- Slope/angle, if θ makes an angle with the line and x axis, the slope can be expressed as: $\tan\theta = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = m$

Secant and Tangent Lines

- A secant line passes through 2 points on the curve and approximates the curve locally.
- For non linear functions secant lines estimate the AROC over an interval
- Tangent Line passes through 1 point and gives a linear approximation at that point.
- Tangent lines give the IROC at a specific point

Graphs and curves

- A graph is a set of points (x, y), satisfying $y = f(x)$.
- A curve is a more general set of points $h(x, y) = 0$.

(Not every curve is graph)

- Graph must pass the vertical line test. (circle $\neq$ graph)

Multivariable Functions

- $z = \sqrt{x^2+y^2}$, Distance formula of (x,y) from the origin to the Cartesian plane.

Things to remember

- The derivative of $f(x)$, repelesents the equation of the slope of the tangent line.

- graphs describe single valued functions, while surfaces can reperesent double valued/complex functions.

EX #1

Find tangent line to $f(x) = 3x^2$ @ $(2, 1)$.

$f'(x) = 6x$

$m = 6(2)$

$m = 12$

$y = mx + b$

$1 = 2(12) + b$

$1 = 24 + b$

$b = -23$

$\boxed{y = 12x - 23}$

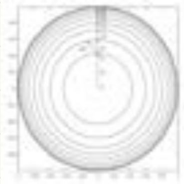
Lecture 3 Notes

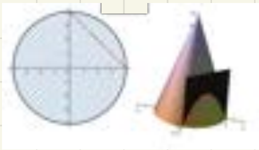
Countour Diagrams

- Reperesent lines of constant values for $z = f(x,y)$ at different heights of $z = C$.
- Each contour reperesents a circle centered at origin with radius $\sqrt{C}$,
- when lines are closer together, they have steeper slope.

Examples of Contour Diagrams

- Cone: Concentric circles evenly spaced
- Hemisphere: Contours get closer to origin as you approach centre of hemisphere





- Elliptic paraboloid: Contours are circles that are more spred out as they move away from the centre.
- Bell shaped: The contours spread near their max, then get closer as they descend from the center.

Cross Section

- A cross section is a 2D slice of a 3D object created by fixing one variable $x = C$
- Example, For a sphere $x^2+y^2+r^2=1$ a cross section parallel to the YZ plane woud give circle with radius of $\sqrt{1-c^2}$ within the sphere

Important Surfaces

Cone: $z = \sqrt{x^2+y^2}$

Sphere: $x^2+y^2+z^2=r^2 \longrightarrow (x-h)^2+(y-k)^2+(z-l)^2=r^2 \rightsquigarrow$ Sphere with radius 5 centred at $(2,-1,3)$

$(x-2)^2+(y+1)^2+(z-3)^2=25$

Cylinder: $x^2+y^2=1$ (Straight up from z axis)

$\downarrow$ Whichever variable is not in the equation is where the cylinder lies along.

Hyperbolic Paraboloid: $z = x^2-y^2$

Lecture 4 Notes

Linear Functions

Linear functions extend to multiple variables.

Linear Functions in one Variable

- $y = mx + b$
- Geometrically, reperesents a straight line in a 2D plane.
- Horizontal lines: $y = b$
- Vertical lines: $x = x_0$ (undefined slope)

Linear Functions in 2 variables

- $z = mx + ny + C$
- where m is slope in x direction
  n is slope in y direction
  C is intercept when $x, y = 0$
- This is the Equation of a plane

Any plane can be expressed as
$ax + by + cz = d$

Where a, b, c are directional vectors. d reperesents distance from the origin.

Example 1

Finding Equation of Plane given three Points
$(0,0,0)$ $(2,1,3)$, $(4,2,2)$

1) Get 2 direction vectors $\vec{AB} = (2,1,3)$
$\vec{AC} = (4,2,2)$

2) Cross product, the directional vectors to get normal vector to the plane

$$\begin{matrix}\cancel{2} & 1 & 3 & 2 & 1 & \cancel{3}\\ \cancel{4} & 2 & 2 & 4 & 2 & \cancel{2}\end{matrix}$$

$$= (-4, 8, 0)$$

3) Plug into $ax + by + cz = d$ to find d

$$-4(0) + 8(0) + 0(0) = d$$
$$d = 0$$

$\therefore$ $4x - 8y = 0$ is the equation.

Lecture 5 Notes

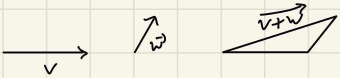
Displacement Vectors

- Defined by 2 points $\overrightarrow{PQ}$
  - Magnitude $|\overrightarrow{PQ}|$, is the distance between the two points
  - Direction: Angles with coordinate axes.

- Equal Vectors: Vectors are equal if they share magnitude, direction.

Vector Operations

1.) Addition and Subtraction: (Tip to tail) $\vec{v} + \vec{w}$



2.) Scalar multiplication: $\| \lambda \vec{v} \| = |\lambda| \, |\vec{v}|$

3.) Any 3D vector can be broken into components: $\vec{v} = (x, y, z)$

Parallel and Perpendicular Vectors

- 2 vectors are perpendicular if their dot product is equal to 0.
- Dot product: $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

or

$$a \cdot b = |a||b| \cos \theta$$

- 2 Vectors are Parallell if they are scalar multiples of each other. This also implies, their dot product is not zero.

$$\underset{\vec{v}}{(1, 3, 2)} \rightarrow \underset{\vec{w}}{(3, 9, 6)} \qquad \vec{w} = 3\vec{v} \quad \therefore \text{ Parallell.}$$

Finding angles between 2 vectors:

$$a \cdot b = |a|\,|b| \cos \theta$$

$$\cos \theta = \frac{a \cdot b}{|a||b|}$$

$$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$$

Magnitude and Direction

- Magnitude: $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

- Unit vector: Normalize a vector to unit length $\vec{u} = \frac{\vec{v}}{||v||}$

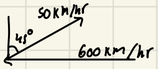
Applications:

- The velocity of a moving object is a vector whose magnitude is the speed of the object and direction = direction.

- Velocity includes direction, speed is magnitude

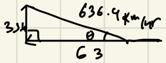
Space for example
6 p730

Example - Plane Example



x	y
600	0
$25\sqrt{2}$	$25\sqrt{2}$

$$50 \cos 45° = 25\sqrt{2}$$
$$50 \sin 45 = 25\sqrt{2}$$
$$\vec{v} + \vec{w} = 600\vec{i} + 35.4\vec{i}$$
$$= 635.4i + 35.4j$$



$\tan\theta$

$$\theta = \tan^{-1}\left(\frac{35.4}{635.4}\right)$$
$$\theta = 3.2°$$

magnitud $= \sqrt{635.4^2 + 35.4^2} = 636.4$

Lecture 6

- Direction is the angle relative to coordinate axes (i, j, k).
- Adding / subtraction: $\vec{v} + \vec{w} = (v_1 + w_1,\ v_2 + w_2,\ v_3 + w_3)$
- Scalar multiplication: Changes direction, if sign flipped.
- Unit vector $\vec{u} = \frac{|\vec{v}|}{||\vec{v}||}$
- $\vec{0}$ vector
- When objects move at constant velocity, its displacement vector is always parallel to its velocity vector.
- Acceleration is specified by magnitude and Direction
- Force is another example of a vector

Examples

Lecture 7

Dot Products and Planes

Geometrically: $\vec{a} \cdot \vec{b} = |a||b|\cos\theta$

Algebraiclly: $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Key Properties:

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

$\vec{v} \cdot (\vec{w} \cdot \vec{v}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$

$\vec{v} \cdot \vec{v} = ||v||^2$

Planes in 3D

Equation of a plane:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Where (x_0, y_0, z_0) is a point on the plane and $\vec{n} = (a, b, c)$ is normal vector

or

$$ax + by + cz = d$$

Examples

Lecture 8 Notes

Projections

- Decomposing $\vec{v}$ into parallel and perpendicular components relative to a vector $\vec{u}$
  - $\vec{v}_{parallel} = (\vec{v}\cdot\vec{u})\vec{u}$
  - $\vec{v}_{perp} = \vec{v} - \vec{v}_{parallel}$
- The projection of v onto u, is the component of v in the direction of $\vec{u}$.
  - Projection of v onto $\vec{u}$ = $\left(\frac{v \cdot u}{u \cdot u}\right)\vec{v}$
- The projection is a vector.

Work and Force

- If wind force F is at 30° to unit vector $\vec{u}$:
  $\vec{F}_{parallel} = \|\vec{F}\|\cos 30^\circ \cdot \vec{u}$
- Work done by force $\vec{F}$ through displacement:

$$W = \vec{F}\cdot\vec{d} = \|\vec{F}\|\,\|\vec{d}\|\cos\theta$$

- Positive work occurs if $0 \le \theta \le \frac{\pi}{2}$
- Negative work occurs if $\frac{\pi}{2} < \theta < \pi$

- Area of parallelogram: Base · Height
  $= \|v\|\,\|w\|\sin\theta$

Note on planes

- Two planes are the same if their equations are scalar multiples of each other.
- Two planes are parallel, if they have the same normal vector.
- Two planes, intersect at a line, if normal vectors are different
- Two planes are perpendicular, if their normal's are perpendicular.

Lecture 9 Notes

Cross Product

- Algebraic Def: $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

- Properties:
  - Antisymmetry $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$
  - Perpendicularity $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$
  - Distributive $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{w} + \vec{u} \times \vec{v}$

Area of Parallelogram: $\text{Area} = \|\vec{v} \times \vec{w}\|$

Volume of Parallelipped $\text{volume} = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

Plane Equation using Cross Product:

Given: P, Q, R

$\vec{n} = \vec{PQ} \times \vec{QR}$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Lecture 10 Notes

Definition: $\int_a^b f(x)dx$

The integral reperesents the area between the curve and the x axis.

Riemann Sum: Approximates area by dividing the intervals into Smaller parts.

The definite integral of a Contious function over a finite interval $[a, b]$ is the limit of Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x_i$$

Geometric: If $f(x) \geq 0$, gives area under the curve. If $f(x) \leq 0$, gives the are under the x axis.

Properties of Definite Integrals

1) constants

$$\int_a^b f(x)dx = \int_a^b c\,dx = c(b-a)$$

2) Even

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$f(-x)=f(x)$

3) odd

$$\int_{-a}^{a} f(x)dx = 0$$

$f(-x)=-f(x)$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F(x)$ such that $F'(x) = f(x)$

Indefinite Integral

$$\int x^3 = \frac{x^4}{4} + C$$

Definite Integral

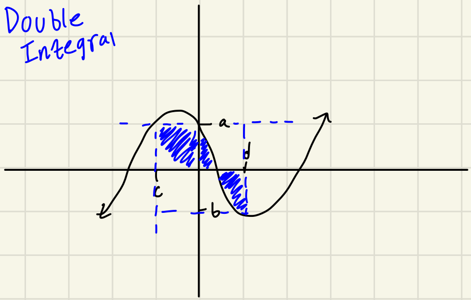
$$\int_0^1 x^2 = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

Even: $f(x) = f(-x)$

Odd: $f(-x) = -f(x)$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$



$$\int_a^b \int_c^d f(x)\,dx\,dy$$

Reperesents the Combined Area or Volume over R

Another way to write double integrals:

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy$$

Special Case:

$$\text{Area of } R = \int_R f \cdot dA = \int_R 1\,dA$$

Extra use cases:

$$\text{Mass} = \iint_R \delta(x,y)\,dA$$

Probability of Landing inside Rectangle $= \iint_R p(x,y)\,dA$

Double Integrals

Def: Extends the idea of area to volume.

For a function $f(x,y)$ defined on a region R, the double calculates the Volume under the surface.

$$\iint_R f(x,y)\,dA$$

Ex

$$\int_0^1 \int_0^1 (x+y)\,dx\,dy$$

1) Solve inner integral with respect to x,
2) Outer integral respect to y.

- The double integral reperesents the area of a sub rectangle bounded by 2 axis,
- Taking the integral with respect to 2 different variables gives us an area
- For rectangular regions, you can swap the order of integration.

One integral gives: 2 dimensional area

Two integrals gives: 3 dimensional volume

Three integrals gives: 4 dimensional hypervolume

Reminder Continuity

- A function is continous on R if:
  - It is defined for every point on R
  - No jumps, holes or ifinite behaviour
  - Limits approach same point on both sides

Lecture 11 Notes

Review: a single variable integral, $\int_a^b f(x)\,dx$ finds the area underneath the curve $f(x)$ from $x=a$ to $x=b$.

Important Properties for single Integrals

1) Linearity:

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2) Homogeneity

$$\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$$

where λ is a constant.

3) Additivity:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Definition of Double Integral:

For a function $f(x,y)$ over a rectangular region R: $a \le x \le b$ and $c \le y \le d$. we have R divided into small rectangle. Each has the Area given by: $\Delta A = \Delta x \cdot \Delta y$

Simplified Double Integrals

By using iterated integrals:

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

First integrate with respect to x
Then integrate with respect to y.

Note: order of integration does not matter.

Probability

If $P(x,y)$ is the PDF, then double integral gives probability of random event falling in R.

$$= \iint_R P(x,y)\,dA$$

Note: use U substitution for terms like $(ax+b)^n$

1) Find $u = ax+b$
Find $du = a$
Isolate for dx

2) Change limits accordingly

3) Integrate using power rule

4) Plug in limits

Lecture 12

Double Integrals over Non-Rectangular Regions

For Rectangular regions R, the limits are constants.

Non-rectangular regions have limits that vary based on x or y.

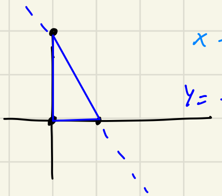
Extended Function: $\iint_R f(x,y)\,dA = \iint_{R1} f_1(x,y)\,dA$

where $f_1(x,y) = f(x,y)$ inside R and 0 outside

Example: Triangle Plate

Vertices at $(0,0)$, $(1,0)$ and $(0,2)$
with density function $\delta(x,y)$

Plate is bounded by: $Y = 2-2x$
or
$x = 1 - \frac{y}{2}$



$Y = -2x + 2$

Y runs from $y=0$ to $y = 2-2x$

Because the vertical strip starts at $(x,0)$ ends with the point $(x, 2-2x)$

$$\int_0^{2-2x} \delta(x,y)\,dy$$

Then for each x is between 0, 1

Therefore, Mass $= \int_0^1 \int_0^{2-2x} \delta(x,y)\,dy\,dx$

For Iterated Integrals

- limits for outer integral must be constants
- limits for inner integral's must involve only the variable used in outer integration

Example: Bounded Metal Plate

- Plate is bounded by $y = x$ and $y = x^2$ with density $\delta(x,y) = 1 + xy$

$$\text{Mass} = \int_0^1 \int_{x^2}^{x} 1 + xy\,dy\,dx$$

$$\int_{x^2}^{x} 1 + xy\,dy = y + \frac{xy^2}{2}\Bigg|_{y=x^2}^{y=x}$$

$$= x^2 + \frac{x^5}{2} - x + \frac{x^3}{2}$$

$$\int_0^1 x^2 + \frac{x^5}{2} - x + \frac{x^3}{2}$$

$$= \frac{x^3}{3} + \frac{x^6}{2 \cdot 6} - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4}\Bigg|_0^1$$

$$= \left(\frac{1}{2} + \frac{1}{8}\right) - \left(\frac{1}{3} + \frac{1}{12}\right) - 0$$

$$= \frac{5}{24}$$

Example 5:

- City has semicircular region of radius 3 km. Find distance from points in city to ocean

Circle: $x^2+y^2=9$

Average Distance $= \frac{1}{\text{Area}(R)} = \int_R y\,dA$

Area of a half circle: $\frac{9\pi}{2}$

$x^2+y^2=9 \to y=\sqrt{9-x^2}$

Since there's a strip for every x from -3 to 3

we get:

$$= \int_R y\,dA = \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} y\,dy\,dx$$

$$\int_{0}^{\sqrt{9-x^2}} y\,dy = \frac{y^2}{2}\Big|^{y=\sqrt{9-x^2}}$$

$$\int_{-3}^{3}\frac{1}{2}(9-x^2)\,dx$$

$$= \int_0^3 (9-x^2)\,dx$$

$$= 9x-\frac{x^3}{3}\Big|_0^3 = 18$$

Reversing the order of Integration

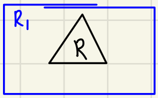
- Reversing can simplify integrals

Ex: $\int_0^6\int_{\frac{x}{3}}^{2} x\sqrt{y^3+1}\,dy\,dx$

$$I = \int_0^2\int_0^{3y} x\sqrt{y^3+1}\,dx\,dy$$

Extended Function

- When we want to integrate a function over an irregular region R, we fit R into a rectangular region R_1.



$F_1(x,y)$ extends $f(x,y)$ to the whole rectangle R1 but turns off R

$$f_1(x,y) = \begin{cases} f(x,y) & \in R \\ 0 & \notin R \end{cases}$$

Basically an equivalent integral that isolates volume over R. Lets us extend the function over an easier shape R1.

Simple Rectangular Regions

- A region R, is rectangular if x and y are bounded independently:

Ex: $\{(x,y): 0 \le x \le 3,\ 1 \le y \le 4\}$

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

How to Solve Non-Rectangular Region Problems

Def: A region R is non rectangular, if x and y bounds depend on each other.

Double integral becomes: $\iint_R f(x,y)\,dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\,dy\,dx$

① Sketch the region R and identify bounds

② Decide on an order of integration:

Vertical slices: - x is fixed
- y varies with respect to x $dy\,dx$

Horizontal slices: - y is fixed
- x varies with respect to y $dx\,dy$

③ Set up integral / solve as iterated integral

Lecture 13

Double Integrals in Polar Coordinates

- For circular or radial regions, using Polar Coordinates (r, θ) instead of Cartesian (x,y) simplifies the setup.
  - r is the distance from the origin
  - θ is the angle from the positive x axis

Converting From Cartesian to Polar Coordinates:

$$x = r\cos\theta \,, \quad y = r\sin\theta \quad \text{and} \quad x^2+y^2=r^2$$

Converting Formula: $f(x,y) = f(r\cos\theta, r\sin\theta) = f_1(r,\theta)$

$$0 \le r < \infty \quad \text{and} \quad 0 \le \theta \le 2\pi$$

Integral's in Polar Coordinates

To integrate a function $f(x,y)$ over a region R, which we convert $f(x,y)$ to $f(r,\theta)$:

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot \boxed{\underline{r}\,dr\,d\theta}$$

This extra term is called the Jacobian and accounts for the change in area from Cartesion to P.C

Jacobian

Factor that corrects for how stretched/compressed space becomes when changing coordinates.

Must be used when converting coordinate systems.

Example:

For circular ring, $R: 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{4}$

with $f(x,y) = \frac{1}{(x^2+y^2)^{3/2}}$

Solution:

Convert $f(x,y)$ into polar form $f(x,y) = \frac{1}{r^3}$

$$\iint_R \frac{1}{r^3} \cdot r \, dr \, d\theta = \int_0^{\frac{\pi}{4}} \int_1^2 \frac{1}{r^2} \, dr \, d\theta$$

- In polar coordinates circular regions simply, become rectangles.

Triple Integrals

- Triple integrals extend the idea of double integrals, to three dimensions, allowing us to calculate volumes, masses or averages over 3D regions.

$$\iiint_W f(x,y,z) \, dV \longrightarrow \underbrace{\int_p^q \int_c^d \int_a^b f(x,y,z) \, dx \, dy \, dz}_{\text{Can choose any order of integration.}}$$

Riemann Sum for Volumes

- Break's down 3D region into Small boxes: $\Delta V = \Delta x \, \Delta y \, \Delta z$

- Sum the values f in each box, then take limit as approaches 0

Rules for Triple Integrals

- The outer integral has constant limits
- The middle integral. Can involve only the variable (that is in the outer integral)
- The inner integral integral can involve the two variables (those on outer 2 integrals)

How to Solve Volume / Triple integral Questions

① Visualize the bounds in the coordinate plane

② For triple integrals, set a variable to 0 and see if the variable depends on the other variable.

③ Set Bounds carefully (the last integral should be a constant)

How to solve converting integrals to Polar Coordinates

① vizualize given bounds and their shape

② Replance any $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$

③ Jacobian: $r\,dr\,d\theta$

④ Find radius from origin to boundary, θ angular span

Example: Mass of Cube:

For a cube with density: $\delta(x,y,z) = 1 + xyz$
To find the mass of the piece, we do:

Mass of Piece $\approx$ Density · Volume $\approx \delta(x,y,z)\Delta V$

Take the sum of the masses as limit of $\Delta V \to 0$.

$$\text{Mass} = \int_C \delta\, dV = \int_0^4\int_0^4\int_0^4 (1+xyz)\,dx\,dy\,dz$$

$$= \int_0^4 (1+xyz)\,dx$$

$$x + \frac{x^2}{2}yz \Big|_0^4$$

$$= 4 + 8yz$$

$$\int_0^4 4 + 8yz\,dy$$

$$= 4y + 4y^2 z \Big|_0^4 = 16 + 64z\,dz$$

$$\int_0^4 16 + 64z\,dz = 16z + 32z^2 \Big|_0^4$$

$$= 576\text{ gm}$$

Example: Volume of Building

Given a buildings slanted roof with an equation $z = 12 - \frac{x}{4} - \frac{y}{8}$

$$\text{Volume} = \int_0^8\int_0^{16}\int_0^{12-\frac{x}{4}-\frac{y}{8}} 12 - \frac{1}{4}x - \frac{1}{8}y\,dz\,dy\,dx$$

1) $$\int_0^{12-\frac{x}{4}-\frac{y}{8}} 12 - \frac{1}{4}x - \frac{1}{8}y\,dz$$

$$= 12z - \frac{1}{4}xz - \frac{1}{8}yz \Big|_0^{12}$$

$$= \int_0^{16} 144 - 3x - \frac{3}{2}y\,dy$$

$$= 144y - 3xy - \frac{\frac{3}{2}y^2}{2} \Big|_0^{16}$$

$$= 2304 - 48x - 192$$

$$\int_0^8 2304 - 48x - 192\,dx$$

$$= 2304x - \frac{48x^2}{2} - 192x \Big|_0^8$$

$$= 15360\text{ m}^3$$

Polar Coordinates - Cartesian Relations

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (depending on quadrant) add $\pi, \pi, 2\pi$ accordingly

Polar coordinates can be expressed infinitly many ways by adding 2π or changing direction.

If $r =$ negative, line goes opposite way from angle.
If $r =$ positive, line goes from x axis to line.

Polar coordinates make it easier to work with circles

Lecture 14 Notes

Polar Coordinates (2D)

Each point (x, y) in 2D is described by (r, θ).

r is the distance from the origin

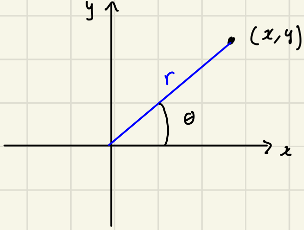
θ is the angle from positive x axis.

Conversions between Cartesian and Polar:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$$

They simplify integrals for regions with circular symmetry.

Polar coordinates: (r, θ)



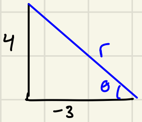
Converting Rectangular to Polar

$(2, -4) \rightarrow (r, \theta)$ $\quad x = r\cos\theta, \; y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$ $\qquad \theta = \tan^{-1}\left(\frac{4}{2}\right) \longrightarrow \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{2^2 + 4^2}$ $\qquad \theta = 63.435$

$r = 2\sqrt{5}$ $\qquad (2\sqrt{5}, 63.485°)$



$x = r\cos\theta$

$y = r\sin\theta$

$-3 = r\cos\theta$

$4 = r\sin\theta$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\qquad r = \sqrt{3^2 + 4^2}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$ $\qquad r = 5$

$\theta = 53°$ $\qquad 180 - 53 = 127°$

Cylindrical Coordinates

- Cylindrical coordinates (r, θ, z) extend Polar coordinates into 3D by adding a vertical z coordinate

Basically Polar Coordinates in 3 Dimensions.

Cartesian to Cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r = \sqrt{x^2 + y^2}$

where:

$0 \leq r < \infty$

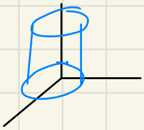
$0 \leq \theta \leq 2\pi$

$-\infty < z < \infty$

Fundamental Surfaces

(c is a constant)

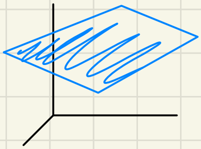
$r = c$: A cylinder around the z axis



$\theta = c$: A half plane through the z axis

(z is fixed)

$z = c$: A horizontal plane



Triple Integrals in Cylindrical Coordinates

To integrate over cylindrical regions W, we use

$$\int_W f(x,y,z)\,dV = \iiint f(r\cos\theta, r\sin\theta, z)\cdot r\cdot dr\cdot d\theta\cdot dz$$

Ex2

$$\int_0^4\int_0^{\frac{\pi}{6}}\int_0^6 1.2r\cdot dr\,d\theta\,dz$$

$$\int_0^6 1.2r\,dr = \frac{1.2r^2}{2}\Big|_0^6 = 21.6$$

$$\int_0^{\frac{\pi}{6}} 21.6\,d\theta = \frac{21.6\,\theta}{1}\Big|_0^{\frac{\pi}{6}} = \frac{18}{5}\pi$$

$$\int_0^4 \frac{18}{5}\pi\,dz = \frac{18}{5}\pi\cdot z\Big|_0^4 = \frac{72}{5}\pi = 45.240 \text{ grams}$$

Spherical Coordinates (3D)

- Spherical coordinates (ρ, ϕ, θ) describe points by:
  - ρ: Distance from the origin
  - ϕ: Angle from positive z axis (elevation angle)
  - θ: Angle from positive x axis in xy plane (like polar coordinates)

$\rho^2 \sin\phi$ is the Jacobian

θ is always angle from positive x axis.

Conversions from Cartesian to Spherical:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\theta\sin\theta$$
$$z = \rho\cos\phi$$

with $\rho = \sqrt{x^2+y^2+z^2}$

$$\rho^2 = x^2+y^2+z^2 = r^2+z^2$$

$$r = \rho\sin\phi$$

Relations

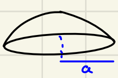
$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\cos\theta$$
$$z = \rho\cos\phi$$

Rectangle: (x, y)
Polar: (r, θ)
Cylindrical: (r, θ, z)
Spherical: (ρ, ϕ, θ)

$$x = \rho\sin\phi\cos\phi$$
$$y = \rho\sin\phi\sin\phi$$
$$z = \rho\cos\phi$$
$$\rho = \sqrt{x^2+y^2+z^2}$$

Example: Volume of Water in Hemisphere

Hemisphere: $x^2+y^2+z^2=a^2$



In cylindrical coordinates this becomes $r^2=z^2+a^2$

$$r^2+z^2=a^2$$

Setting limits:

- Water goes up to depth h, z goes from 0 to h.
- θ goes from 0 to 2π
- r at each height goes from 0 to $\sqrt{a^2-z^2}$.

$$V=\iiint_W 1\,dV$$

$$V=\int_0^{2\pi}\int_0^{h}\int_0^{\sqrt{a^2-h^2}} 1\cdot r\,dr\cdot dz\cdot d\theta$$

$$\int_0^{\sqrt{a^2-h^2}} 1\cdot r\,dr = \frac{r^2}{2}\Big|^{\sqrt{a^2-h^2}}$$

$$\int_0^{h}\frac{1}{2}(a^2-z^2)\,dz = \frac{1}{2}\left(a^2z-\frac{z^3}{3}\right)\Big|_{z=0}^{z=h}$$

$$d\theta=\int_0^{2\pi}\frac{1}{2}\left(a^2h-\frac{h^3}{3}\right)d\theta$$

$$=\pi\left(a^2h-\frac{h^3}{3}\right)$$

Example 2: Mass of Solid

Bounded by $z=\sqrt{x^2+y^2}$ and $z=3$ with a density function $\delta(x,y,z)=z$

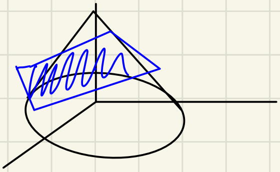
$r=\sqrt{x^2+y^2}$ and $dV=r\cdot dr\cdot d\theta\,dz$

The cone $z=\sqrt{x^2+y^2}$ becomes $z=r$

Limits: $0\le r\le 3$

$0\le\theta\le 2\pi$

$r\le z\le 3$



$$\iiint_W z\,dV=\int_0^{2\pi}\int_0^{3}\int_r^{3} z\cdot r\cdot dz\,dr\,d\theta$$

$$\int_r^{3} z\cdot r\,dz = \frac{z^2\cdot r}{2}\Big|_r^{3} = \frac{9-r^2}{2}$$

$$\int_0^{3}\frac{9-r^2}{2}\,dr = \frac{1}{2}\left(9r-\frac{r^3}{3}\right)\Big|_0^{3}$$

$$=\int_0^{2\pi}\frac{9r^2}{4}-\frac{r^4}{8}\Big|_0^{3} = \int_0^{2\pi}\frac{81}{8}\,d\theta$$

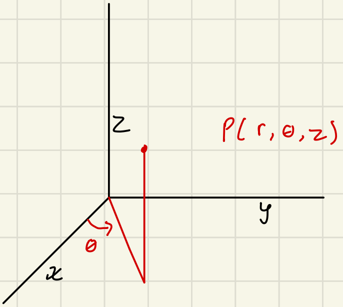
$$M=\frac{81}{8}\cdot 2\pi = 63.6172$$

Lecture 15

Cylindrical Coordinates

For a point (x, y, z) in Cartesian Coordinates,
Cylindrical coordinates are (r, θ, z).

- r, the distance from the origin in the xy plane
- θ, the angle from positive x axis
- z, Same as z in C.C



Conversions:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Spherical Coordinates (3D)

- Spherical coordinates (ρ, ϕ, θ) describe points by:
  - ρ: Distance from the origin
  - ϕ: Angle from Positive z axis (elevation angle)
  - θ: Angle from positive x axis in xy Plane (like Polar Coordinates)

$\rho \geq 0$

$0 \leq \phi \leq \pi$

(elevation angle goes max 180°)

Conversions from Cartesian to Spherical:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

with $\rho = \sqrt{x^2 + y^2 + z^2}$

The volume element dV in spherical coordinates becomes: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Converting Coordinate Systems

Cylindrical ⟶ Rectangle

$P(r,\theta,z) \rightarrow P(x,y,z)$

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

Rectangular → Cylindrical

$P(x,y,z) \rightarrow P(r,\theta,z)$

$r^2 = x^2+y^2 \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \quad z = z$

Cartesian → Spherical

$P(x,y,z) \rightarrow P(\rho,\phi,\theta)$

$\rho = \sqrt{x^2+y^2+z^2}$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$\rho^2 = x^2+y^2+z^2 = r^2+z^2$

Cylindrical → Spherical

$\rho = \sqrt{r^2+z^2}$

$\phi = \tan^{-1}\left(\frac{z}{r}\right)$

$\theta = \theta$

Spherical → Cylindrical

$r = \rho\sin\phi$

$z = \rho\cos\phi$

$\theta = \theta$

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = z = \rho\cos\phi$

Example: Integration in Cylindrical Coordinates

Find the integral:

$$\int_W 2xyz\,dx\,dy\,dz$$

where W is a cylinder of radius 1 and height 1.

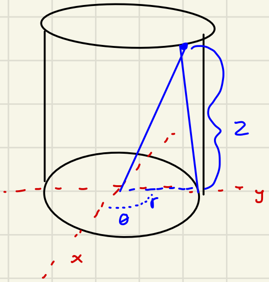
$$r^2 = x^2+y^2 \le 1, \quad 0 \le z \le 1.$$

We know: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$2xyz = 2\cdot(r\cos\theta)\cdot(r\sin\theta)\cdot z = \underline{2r^2 z\cos\theta\sin\theta}$$

using $2\cos\theta\sin\theta = \sin 2\theta$

we get: $2xyz = r^2 z\sin 2\theta$



Default limits:

$$\int_0^1\int_0^1\int_0^{2\pi} r^2\cdot z\cdot\sin 2\theta\,d\theta\,dr\,dz$$

We know:

$$\int_0^{2\pi}\sin 2\theta\,d\theta = 0$$

$\therefore$ We conclude $I = 0$

Example: Integration in Spherical Coordinates

- Use Spherical Coordinates to derive the formula for the volume of a ball of radius a.

In general we:

$$f(x,y,z) = f_1(\rho,\phi,\theta)$$

$$\int_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W1} f_1(\rho,\phi,\theta)\underbrace{\rho^2\sin\phi}_{\text{Jacobian}}\,d\rho\,d\phi\,d\theta$$

For Question:

$$0 \le \rho \le a, \quad 0 \le \theta \le 2\pi \text{ and } 0 \le \phi \le \pi$$

$$\text{Volume} = \int_W 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^a \rho^2\,d\rho = \frac{\rho^3}{3}\Big|_0^a = \frac{a^3}{3}$$

$$\int_0^{\pi}\sin\phi\,d\theta = [-\cos\phi]_0^{\pi} = 2$$

$$\int_0^{2\pi} d\theta = 2\pi \longrightarrow \text{Volume} = \frac{2a^3}{3}\cdot 2\pi$$

$$= \frac{4}{3}\pi a^3$$

Trig Identities

$\cos^2\theta + \sin^2\theta = 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

$\sin 2\theta = 2\sin\theta\cos\theta$

10) $f(x, y, z) = x^2+y^2+z^2$

1) convert to (r, θ, z)

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$x^2+y^2 = r^2$

This gives us $= r^2+z^2$

$$\int_{-1}^{1}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{0}^{4} r^2+z^2 \cdot r\,dr\,d\theta\,dz$$

$$\int_{0}^{4} r^3+z^2r\,dr = \frac{r^4}{4}+\frac{z^2r^2}{2}\Big|_{0}^{4}$$

$= 64+8z^2$

$$64+8z^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta = \frac{3\pi}{4}-\frac{\pi}{4} = \frac{1}{2}\pi$$

$$64\cdot\frac{1}{2\pi}\int_{-1}^{1} 8z^2\,dz = \frac{8z^3}{3}\Big|_{-1}^{1} = \frac{8}{3} - -\frac{8}{3}$$

$= 64\cdot\frac{1}{2\pi}\cdot\frac{16}{3} = \frac{512\pi}{3}$

Lecture 16 Notes

Differentiation of Single-Variable Functions

Def: The derivative $f'(a)$ of a function $f(x)$ at $x=a$ is given by: $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

To be a differentiable function means the derivative exists at every point in the domain. So bassically a valid slope.

Linear Approximation:

A differentiable function can be approx. by its tangent line near a point $f(x) \approx f(a) + f'(a)(x-a)$ → Gives a linear approximation

Convexity and Concavity

$f(x)$ is concave up if $f''(x) > 0$ ∪ shape

$f(x)$ is concave down if $f''(x) < 0$ ∩ shape

Inflection Points - If $(c, f(c))$ is a point of inflection of the graph of f, then $f''(c) = 0$ or f'' does not exist at c.

Differentiation of Multivariable Functions

- For a function $z = f(x,y)$, partial derivatives describe how z changes as x varies or y varies, holding the other constant.

Partial Derivatives

$$x: \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}$$

$$y: \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}$$

Notation $\frac{\partial f}{\partial x} = f_x = \partial_x f$

Note: Partial Derivatives reperesent the rate of change along a variable.

Gradient: Made up of all partial derivatives.

Examples of Partial Derivatives

1.) $f(x,y) = y^2 \cdot e^{3x}$

$f_x = e^{3x} \cdot 3 \cdot y^2 = y^2 \cdot 3e^{3x}$

$f_y = 2y \cdot e^{3x}$

2.) $z = (3xy + 2x)^5$

$f_x = 5(3xy + \underline{2x})^4 (3y+2)$ → Note this beomes 0.

$f_y = 5(3x)(3xy+2x)^4$

3) $g(x,y) = e^{x+3y} \cdot \sin(x,y)$

$g_x = e^{x+3y} \cdot \sin(x,y) + e^{x+3y} \cdot \cos(x,y) \cdot y$

Tangent Plane and Local Linearization

- For a differentiable $f(x,y)$ near (a,b)

$$f(x,y) \simeq f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Tangent Plane Equation: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

The tangent line is the Same concept in 2 dimensions → Equation of tangent

Tangent line to function, $y = f(x)$ at Point $x = a$, touches point a and has same Slope at that point.

$$y - f(a) = f'(a)(x-a)$$

Example:

1) Equation for tangent Plane to $z = \sin(x^2y)$ at the Point $(5,0,2)$.

$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

$f_x = \cos(x^2y)(2xy) \xrightarrow[x=5,\,y=0]{} f_x(5,0) = 0$

$f_y = \cos(x^2y)\cdot x^2 \longrightarrow f_y(5,0) = 25$

$z = 0\cdot(x-5) + 25\cdot(y-0) + 2$

$z = 25y + 2$

Lecture 17 Notes

Partial Derivatives measure the rate of change of f along the x axis or the y axis.

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}, \quad f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Directional Derivatives

Def: The directional derivative $f_u(a,b)$, gives the rate of change of f at (a,b) in the direction of unit vector $u = (u_1, u_2)$

If $u = i$ or $u = j$, then f_u simplifies to f_x or f_y.

For any non unit vector use $\vec{U} = \frac{V}{\|V\|}$

Example: For $f(x,y) = x^2 + y^2$ at $(1,0)$ in direction of $V = i + j$

unit vector : $\vec{U} = \frac{1}{\sqrt{2}}(i + j) \longrightarrow$ Find unit vector

Directional derivative :

$$\nabla f = 2x, 2y$$
$$2, 0$$

$$(2,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$D_u F = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \cdot \vec{U}$$

at (a,b)

Gradient Vector

The gradient vector of f at (a,b) denoted $\nabla f(a,b)$ is :

$$\nabla f(a,b) = f_x(a,b)\, i + f_y(a,b)\, j$$

The gradient vector points in the direction of the steepest ascent of f and its magnitude $\|\nabla f(a,b)\|$ gives the rate of that increase. → Slowest means opposite direction.

Relationship between Gradient and Directional Derivative

The directional derivative : $f_u(a,b) = \nabla f(a,b) \cdot \vec{U}$

Note: f increases fastest in the direction of ∇f

f decreases fastest opposite direction to ∇f

f does not change in directions perpendicular to ∇f.

Lecture 18 Notes

Gradient Vector in 2D

- The gradient of a function $f(x,y)$ at a point (a,b) is a vector.

$$\nabla f(a,b) = f_x(a,b)i + f_y(a,b)j$$

Key Takeaway: The gradient points in the direction of the steepest increase of f.

Directional Derivatives in 2D

- The directional derivative of $f(x,y)$ at (a,b) in the direction of unit vector $u = u_1 i + u_2 j$ measures the rate of change of f in that direction.

$$f_u(a,b) = \nabla f(a,b) \cdot \vec{u} \rightarrow$$

is a finite number, tells you how fast f is changing in the direction $\vec{u}$.

Geometrically: $f_u(a,b)$ is maximized when u is aligned with ∇f

Max Rate of Change: $\|\nabla f(a,b)\|$

Min Rate of Change: $-\|\nabla f(a,b)\|$ (Opposite Direction)

Tangent Lines to Level Curves

- A level curve is where $f(x,y) = C$ for a constant
- The gradient $\nabla f(a,b)$ is perpendicular to the level curve at (a,b).

- The equation of the tangent line to the level curve at (a,b) is:

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) = 0$$

Note: Gradient vector is perpendicular to level surfaces. Equation of tangent becomes gradient with points plugged in times normal; $(x-a)$ $(y-b)$ $(z-c)$

Gradients and Directional Derivatives in 3D

- For $f(x,y,z)$ the gradient is:

$$\nabla f(a,b,c) = f_x(a,b,c)i + f_y(a,b,c)j + f_z(a,b,z)K$$

- Directional Derivative:

$$f_v(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$$

Tangent Planes to Level Surfaces (3D version, indentical to 2D)

- A level surface is where $f(x,y,z) = C$ for a constant C.
- The gradrent $\nabla f(a,b,c)$ is perpendicular to the surface at (a,b,c).
- The equation of the tangent plane at (a,b,c) is:

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

Example: Tangent Line to Circle

Given: $f(x,y) = x^2+y^2$, level curve $f(x,y) = 1$ is a circle

Solve $\nabla f(x,y) = f_x + f_y$

$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$

At (x_0, y_0), The tangent line is: $2x_0(x-x_0) + 2y_0(y-y_0) = 0$

Example: Finding Directional Derivative of $3xy^3 - 2xz^2$ in the direction $\vec{v} = (2,-3,6)$.

1) $f_u(x,y,z) = ?$

$f_u(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$

$\nabla f(x,y,z) = f_x(x,y,z) \cdot i + f_y(x,y,z) \cdot j + f_z(x,y,z) \cdot K$

$f(x,y,z) = 3xy^3 - 2xz^2$

$f_x = 3y^3 - 2z^2$

$f_y = 3x(3y^2) = 9xy^2$

$f_z = -2x(2z) = -4xz$

$\vec{u} = \frac{V}{\|V\|}$

$V = (2,3,6)$

$\vec{u} = \frac{(2,3,6)}{7}$

$\vec{u} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

$= (3y^3 - 2z^2) \cdot \frac{2}{7} + 9xy^2\left(\frac{3}{7}\right) + (-4xz)\left(\frac{6}{7}\right)$

$f_u = \frac{6}{7}y^3 - \frac{4}{7}z^2 + \frac{27}{7}xy^2 - \frac{24}{7}xz$

Plug in Point $(3,1,-2)$

$= \frac{53}{7}$

Example HW #8 Q6

$f(x,y) = x^2y$ and $v = (4,-3)$ @ $(2,6)$

$f_u = \nabla f(x,y) \cdot \vec{u}$

1) $\vec{u} = \frac{\vec{V}}{\|V\|}$, $\vec{u} = \left(\frac{4}{5}, -\frac{3}{5}\right)$

2) $f_x = 2x \cdot y$
$f_y = x^2$

3) $f_u = 2xy \cdot \frac{4}{5} + x^2\left(\frac{-3}{5}\right)$

$= \frac{8}{5}xy + -\frac{3x^2}{5}$

4) Plug in Point: $(2,6)$

$= \frac{84}{5}$

Lecture 19 Notes:

Parameterized Curves in 2D

- A curve is parametrized by two functions: $x(t)$ and $y(t)$ → Point: $(x(t), y(t))$

Note: x and y depend on another var t.

Example of Circle: $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$

Properties: The Parameterized curve lands on the circle because $x^2 + y^2 = R^2$

- t reperesents time as it goes on, the curve is traced in Counter Clockwise motion $y(t) = R\sin(\omega t)$ or Clockwise $y(t) = -R\sin(\omega t)$

Periodic Motion:

- The time to complete one full rotation is given by: $T = \frac{2\pi}{\omega}$
- Freauency (rotations per second) = $\frac{1}{T} = \frac{\omega}{2\pi}$

Parametrized Curves in 3D

- Adding third dimension

$$r(t) = \big(x(t), y(t), z(t)\big) = x(t)\,i + y(t)\,j + z(t)\,K$$

Example Helix: $x(t) = R\cos(\omega t)$, $y(t) = R\sin(\omega t)$, $z(t) = t$

The xy trace a Circle, while $z(t)$ increases linearly.

Parametric Equations for Lines

- A line passing through point $P = (x_0, y_0, z_0)$ with direction vector $v = (a, b, c)$ can be parameterized as:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

In vector form:

$$r(t) = \underbrace{r_0}_{\text{Point}} + \underbrace{tv}_{\text{direction Vector}}$$

where $r_0 = x_0 i + y_0 j + z_0 K$

Example: Line Passing between 2 Points

$P = (2, -1, 3)$ and $Q = (-1, 5, 4)$

Direction vector: $PQ = (-3, 6, 1)$

Parametric Equations: $r(t) = \underbrace{(2, -1, 3)}_{\text{Point}} + t\underbrace{(-3, 6, 1)}_{\text{Direction vector}}$

$$\begin{cases} x(t) = 2 - 3t \\ y(t) = -1 + 6t \\ z(t) = 3 + t \end{cases}$$

for t from $0 - 1$

Example: Intersection of Line and Sphere

Find where the line $x = t, y = 2t, z = 1+t$ intersects the sphere $x^2 + y^2 + z^2 = 100$

Sub: $x = t, y = 2t, z = 1+t$ into sphere Eq

$$t^2 + (2t)^2 + (1+t)^2 = 100$$

$$\underbrace{6t^2 + 2t - 99 = 0}$$

$t_1 = -4.23$ $t_2 = 3.90$

∴ The line intersects the plane at 2 points

$(t_1, 2t_1, 1+t_1) = (-4.23, -8.46, -3.23)$

$(t_2, 2t_2, 1+t_2) = (3.90, 7.80, 4.90)$

3. Collision of Particles

- $r_1(t) = (t, 1+2t, 3-2t)$, $r_2(t) = (-2-2t, 1-2t, 1+t)$

To Check Collision:

$$r_1(t) = r_2(t)$$

Set up 3 Equations:

① $t = -2 - 2t \rightarrow 3t = -2$, $t = -\frac{2}{3}$

$1 + 2t = 1 - 2t$

$3 - 2t = 1 + t$

The t does not satisfy all equations.

∴ They do not collide

Check For Intersection

Set $r_1(t) = r_2(s)$

$(1+2t, 3-t, -1+4t) = (3+s, -1+2s, 7+3s)$

3 Equations:

$1+2t=3+s, \quad 3-t=-1+2s, \quad -1+4t=7+3s$

$s = 1+2t-3 = 2t-2$

Sub $s=2t-2$ into ②

$3-t=-1+2(2t-2)$

$3-t=-1+4t-4 \Rightarrow 3-t=-5+4t \Rightarrow 5t=8 \Rightarrow t=\frac{8}{5}$

Sub: $t=\frac{8}{5}$ into $s=2t-2$

$s=2\left(\frac{8}{5}\right)-2 \Rightarrow s=\frac{16}{5}-2 \Rightarrow s=\frac{6}{5}$

$t=\frac{8}{5}$ and $s=\frac{6}{5}$ into $-1+4t=7+3s$

$-1+4\left(\frac{8}{5}\right)=7+3\left(\frac{6}{5}\right)$

$-1+\frac{32}{5}=7+\frac{18}{5}$

$\frac{27}{5}=\frac{27}{5}$

∴ The lines intersect

Parametric Equation of Circles

- A circle of radius R centered at (x_0, y_0) in the x-y plane is given by:

$x = x_0 + R\cos t$ $\quad t\,[0-2\pi]$

$y = y_0 + R\sin t$

x_0, y_0 is just a point

Lecture 20

Parameterized Curves and Motion

- A parameterized curve is defined by a Position Vector:

  $r(t) = (x(t), y(t), z(t))$ dependent on Parameter t.

  This reperesents the motion of an object where t is time, and $r(t)$ gives the Position at t.

Velocity and Acceleration

Velocity ($v(t)$):

Defined as $\frac{dr}{dt}$, reperesenting the IROC of Position.

- Tangent to the curve at $r(t)$: direction of motion.
- Speed $= \| v(t) \|$

Acceleration ($a(t)$)

- Defined as $\frac{dv}{dt} = \frac{d^2r}{dt^2}$, reperesenting the rate of change
- For circular motion, $a(t)$ points towards the circle's center (Centripetal)

Tangent Lines

- The parametric equation of a tangent line to $r(t)$ at t_0:

  $r(t_0) + t\,v(t_0)$

Examples:

Helix: $r(t) = \cos t\, i + \sin t\, j + tK$

Circular Motion:

- Position: $r(t) = (R\cos t, R\sin t)$
- velocity: $v(t) = (-R\sin t, R\cos t)$ tangent to circle
- Acceleration: $a(t) = -r(t)$ perpendicular to velocity

Curve Length

- The length of a curve C, given by $r(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ is:

$$\text{Length of } C = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Ellipse $x = a\cos t$, $y = b\sin t$

Circle : $x = r\cos t$, $y = r\sin t$, $C = 2\pi r$

Vector Valued Functions

- outputs a vector

$$\vec{r}(t) = \langle \underbrace{P(t)}, \underbrace{Q(t)}, \underbrace{R(t)} \rangle$$

Scalar valued components

Some Practise

1) $r(t) = t^2 i + t^3 j + 2t K$

a) Find velocity/acceleration vectors

$V = \frac{dr}{dt} = 2t + 3t^2 + 2$ $\quad \vec{v} = (2t, 3t^2, 2)$

$a = \frac{dr^2}{dt^2} = 6t + 2 \rightarrow 2(3t+1)$ $\quad \vec{a} = (2, 6t, 0)$

$t = -\frac{1}{3}$

b) Tangent line at $t = 1$

$r(t_0) + t v(t_0)$

$r(1) = (1, 1, 2)$

$v(1) = (2, 3, 2)$

$r(t) = (1 + 1 + 2) + t(2 + 3 + 2)$

2) circular Motion

$r = 5$, Parametrized by $r(t) = 5\cos t i + 5 \sin t j$

a) $\frac{dr}{dt} = -5\sin t + 5\cos t$

$\frac{dr^2}{dt^2} = -5\cos t - 5\sin t$

b) $r(t) = (5\cos t, 5\sin t)$, $a(t) = -5r(t)$

Recognizing $a(t)$ is made up of $r(t)$. proves it is pointing to the center.

c) $\|V(t)\| = \sqrt{(-5\sin t)^2 + (5\cos t)^2} = 5$

3) Ellipse length

Circumfrence : $x = 3\cos t$, $y = 2\sin t$

for $0 \le t \le 2\pi$

$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt} = -3\sin t$ $\quad \frac{dy}{dt} = 2\cos t$

$$C = \int_0^{2\pi} \sqrt{9\sin^2 t + 4\cos^2 t}\, dt$$

4) Tangent Line for 3D curve:

$r(t) = \sin t i + \cos t j + t K$

a) Tangent at $t = \frac{\pi}{4}$ $\quad r(t_0) + t v(t_0)$

$r\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\pi}{4} K\right)$ $\quad V = \cos t - \sin t$

$V\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}\right) i - \left(\frac{\sqrt{2}}{2}\right) j + K$

Tangent : $r\left(\frac{\pi}{4}\right) + t \cdot v\left(\frac{\pi}{4}\right)$

5. length of curve

$r(t) = (t^2, t^3)$, $0 \le t \le 1$

$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\|v(t)\| = \sqrt{(2t)^2 + (3t^2)^2}$

$= \sqrt{4t^2 + 9t^4}$

Use U-sub

$$\int_0^1 (4t^2 + 9t^4)^{\frac{1}{2}} dt =$$

6. Uniform Circular motion

$r(t) = 4\cos t i + 4\sin t j$

a) $\|v(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = 4$

$\therefore$ Speed is Constant

b) $a(t) = -4\cos t i - 4\sin t j = -r(t)$

$a(t) \cdot v(t) = 0$

$1 + t = 4 - S$

$2 + 2t = 6 - 3S$

$3 - t = S + 2S$

$1 - 6 = 4 - 2$

$-S = 2$ $\therefore$ NO

$t = 3 - S$

$2 + 2(3 - S) = 6 - 3S$

$2 + 6 - 2S = 6 - 3S$

$S = 2$

$3 - t = S + 4$

$3 - t = 9$

$-t = 6$

$t = -6$

Lecture 21

- The length of a curve c, parameterized as $r(t) = x(t)i + y(t)j + z(t)k$ for t in $[a,b]$ can be found by integrating the speed.

$$\text{Length of } C = \int_a^b \|V(t)\|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Speed is the magnitude of the velocity vector

$$\underbrace{\|V(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{\text{Distance formula for curve}}$$

Example

Ellipse circumfrence:

$$\text{Length} = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

Vector Fields →

- A vector field assigns a vector $v = F(r)$ to each point r in space, it describes quantities such as:

- Velocity of fluid flow
- Force fields (gravity, electricity, magnetism)

Example

Gravitational Field: $F(r) = \frac{GMm}{\|r\|^2} r$

The force points towars origin, and decrease with the square of r.

$F(r) = -yi + xj$

At each point, $F(r)$ is perp to r, repersents circular flow around origin

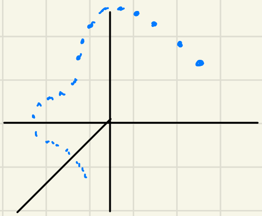
Flow Lines → the trace of the particle affected by vector field

- Flow lines repersent the path traced by particles in a vector field.

Satisfies: $r'(t) = F(r(t))$

where $r(t)$ is the position at time t

$F(r(t))$ is the velocity vector at that point.



Finding Flow Lines

- Break $F(r)$ into components

$F(r) = F_1(x,y)\,i + F_2(x,y)\,j$

Solve the system of differential Equations:

$$x'(t) = F_1(x(t), y(t))$$
$$y'(t) = F_2(x(t), y(t))$$

Example

$F(r) = 3i + 4j$

Flow line passing through $(1,2)$ at $t=0$:

$x(t) = 3t+1,\ y(t) = 4t+2$

Eliminating t gives us: $x = 1,\ y = 2$

Example

Circular Flow:

$F(r) = -yi + xj$

Differential equations: $x'(t) = -y(t),\ y'(t) = x(t)$

Solution: $x(t) = a\cos t,\ y(t) = a\sin t$, describes circles centered at the origin.

$$\underbrace{V(x,y) = -y\,i + x\,j}$$

This equation describes circular motion

Note: Flow Lines reperesent the motion a particle travels while it is in the Vector Field.

Vector Field assigns a Vector to every point in a region.

To Find Flow / Flow Line

1) Break down vector field formula
2) Find $x'(t),\ y'(t)$
3) Integrate with respect to their var's.

More on Vector Fields

- 2D: $F(r) = F_1(x,y)i + F_2(x,y)j$
- 3D: $F(r) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$

Examples

- velocity field
- Force Field

- Gradient vector fields: Forms a vector field that points in direction of steepest increase of f.

Lecture 22

Vector Fields

- A vector field assigns a vector $V(x,y)$ to every point (x,y) in a region.

Def:

$r(t) = (x(t), y(t))$ passes through every point of a region:

$r'(t) = (x'(t), y'(t))$ is a velocity vector, tangent to the curve.

Velocity Vector field: $V(x,y) = (v_1(x,y), v_2(x,y))$

System of Eq for $x(t)$ and $y(t)$:

$$x'(t) = v_1(x(t), y(t))$$

$$y'(t) = v_2(x(t), y(t))$$

Flow Lines

Satisfies the system of differential Equations:

$$x'(t) = v_1(x(t), y(t)), \quad y'(t) = v_2(x(t), y(t))$$

with initial conditions:

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

To Solve flow line questions:

① $x'(t) = x(t)$ Set up

$\frac{dx}{dt} = x$

② $\frac{dx}{x} = dt$ Isolate

③ $\int \frac{dx}{x} = \int dt$ Integrate

$\ln|x| = t$ Find x, then plug in values.

$x = e^t \cdot e^c$

Example of Flow Lines:

$$V(x,y) = i + xj$$

Find Path for Obiect at $(-2, 2)$, $t = 0$

1) $x'(t) = 1$

$\frac{dx}{dt} = 1$

$x(t) = t - 2$

$y'(t) = x(t)$

$y'(t) = t - 2$

$y(t) = \frac{1}{2}t^2 - 2t + 2$

$y(0) = 2$

$y(t) = \frac{1}{2}(t-2)^2$

Flow line: $y = \frac{1}{2}x^2$, parabla passing through $(-2, 2)$

Lecture 23

- A line integral calculates the area under a surface along a particular path within that surface

Geometrically

- A vector field F assigns a vector to every point
- A curve C specifies a path through the field
- The line integral combines vector field with the curve C.

Oriented Curves

- A curve C is oriented, if it has a direction of travel (finite start/end)
- Same curve in reverse is $-C$

Work and Line Integrals

work by a force along straight line:

$$W = F \cdot d = \|F\| \cdot \|d\| \cdot \cos\theta$$

For a constant force F, along displacement d.
where θ is the angle between F and d

when F varies along a curve C:

- Break curve into small displacements

$\Delta r_i = r_{i+1} - r_i$

Total work: $\int_C F \cdot dr$

Calculating Line Integrals

For a curve C,

$$r(t) = (x(t), y(t), z(t)), \quad a \leq t \leq b$$

Line Integral Formula:

$$\int_C F \cdot dx = \int_a^b \underbrace{F(r(t))} \cdot \underbrace{r'(t)}_{\text{Derivative}} dt$$

F(r(t)): Sub the parameterized curve into F.

Example: work Along quarter Circle

Find $\int_C F \cdot dr$ where $F = (x+y)\,i + y\,j$, C is the quarter circle in the first quadrant of radius 1, oriented counterclockwise

1.) parameterize

$$r(t) = \cos t\; i + \sin t\; j \;, \quad 0 \le t \le \frac{\pi}{2}$$

2) Find $r'(t) = (-\sin t\; i) + (\cos t\; j)$

3) subbing $r(t)$ into F:

$$F(r(t)) = (x+y)\,i + y\,j = (\cos t + \sin t)\,i + (\sin t)\,j$$

4) Dot Product $F(r(t)) \cdot r'(t)$:

$$= (\cos t + \sin t)(-\sin t) + (\sin t)(\cos t)$$

$$= -\cos t \sin t - \sin^2 t + \sin t \cos t = -\sin^2 t$$

5) Integral + Solve

$$\int_C F \cdot dr = \int_0^{\frac{\pi}{2}} -\sin^2 t \, dt$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos(2t))\, dt$$

$$= -\frac{1}{2} \left[t - \frac{\sin(2t)}{2} \right]_0^{\pi/2}$$

$$= -\frac{1}{2} \cdot \frac{\pi}{2} = -\frac{\pi}{4}$$

$$\int_C F \cdot dr = -\frac{\pi}{4}$$

Types of Line Integrals

① work done by a force: (treat as normal line integral)

② Line integrals over piecewise paths (solve seperately for each line segment then add them)

③ Closed curve, the line integral is 0. If its conservative

Lecture 24

- Line integrals compute Quantities like work done by a vector field F along a curve C.

Conceptually

- A vector field F(x, y, z) assigns a vector to each point
- A curve C, defines the path
- The line integral, adds up the contribution of F along the curve

Formula: $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$

Key Properties of Line Integrals

1. Reversing orientation:

$$\int_{-C} F \cdot dr = -\int_C F \cdot dr$$

2. Additivity:

$$\int_C F \cdot dr = \int_{C1} F \cdot dr + \int_{C2} F \cdot dr$$

3. closed:

- For a closed curve C, this can be said as the circulation of F around C.

Geometrically / Physical

- Line integrals often reperesent work done, by a force F, along curve
- only the component of F, tangent to the curve contributes to the work
- Circulation around a curve, measures vector Field

- Note: When the vector field is perpendicular to the diameter, the work done is 0.

Setting up Bounds

For line segments

$t = 0$ - start, $t = 1$ - end point

For curves

Semicircles use $t \in [0, \pi]$

Full circle use $t \in [0, 2\pi]$

If curve is reversed $\int_0^1 c = -\int_0^1 c$

Example 1

$\int_c F\cdot dr$, where $F=(x+y)i+yj$ and c is from $P(1,0)$ to $Q(0,2)$ $PQ=(-1,2)$

Parameterization

$r(t)=$ Point $+$ t direction Vector

$r(t)=(1,0)+t(-1,2)$

$=(1,0)+(-t,2t)$

$x=1-t$

$y=2t$

1) $r(t)=(x(t),y(t))$

$r(t)=(1-t)i+2tj$

2) $r'(t)=-i+2j$

3) $F(r(t))=((1-t)+2t)i+(2t)j$

4) $F(r(t))\cdot r'(t)=((1-t)+2t)(-1)+(2t)(2)$

$=-1+t+4t=5t-1$

5.) Integrate over t

$$\int_c F\cdot dx=\int_0^1(5t-1)\,dt$$

$$=\frac{5t^2}{2}-t\Big|_0^1$$

$$=\left[\frac{5}{2}-1\right]-0$$

$$=\frac{3}{2}$$

Example 2: Work along Parabola

$\int_c F\cdot dr$ where $F=(x+y)i+yj$ and c is Parabola: $y=2(1-x^2)$ from $P(1,0)$ to $Q(0,2)$

1) Parameterize: $x=t$

$y=2(1-t^2)$, $0\le t\le 1$

2) $r(t)=(t+2(1-t^2))i+2(1-t^2)j$

$r(t)=ti+2(1-t^2)j \longrightarrow r(t)=(x(t),y(t))$

$r'(t)=i-4tj$

3) Sub into F

$F(r(t))=(t+2(1-t^2))i+2(1-t^2)j$

$=8t^3-2t^2-7t+2$

4) Integrate:

$$\int_c F\cdot dr=\int_0^1 -7t+8t^3-2t^2+2$$

$$=\frac{-7t^2}{2}+\frac{8t^4}{4}-\frac{2t^3}{3}+2t\Big|_0^1$$

$$=\frac{3}{2}$$

Lecture 25

Gradient Vector Fields

- A gradient Vector field is derived from the gradient of a Scalar function $f(x,y,z)$:

$$F = \nabla f \quad \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial r}{\partial z} k$$

- $f(x,y,z)$ is called the potential function of F.

Line Integrals of Gradient Vector Fields

- For a gradient vector field $F = \nabla f$ over a curve C:

$$\int_C F \cdot dr = f(Q) - f(P) \quad \Big\}$$

Fundamental theorem
of Calculus
for line integrals

where:

- P and Q are the Starting/ending Points of C.
- The integral is dependent only on P and Q, not the Path C.

Key Property:

- Gradient vector fields are Path-independent, meaning the result is the same for any smooth curve C between P and Q.

Conservative Vector Field

A Vector field is: Conservative

1.) It is the gradient of Some Potential function $f(x,y,z)$, $F = \nabla f$

2.) The line integral $\int_C F \cdot dr$ is Path independent

Vector Field is Conservative?

is $F(x,y) = (2xy + y^2)i + (x^2 + 2xy)j$ is conservative

1.) check: $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$:

$F_1(x,y) = 2xy + y^2$, so

$$\frac{\partial F_1}{\partial y} = 2x + 2y$$

$F_2(x,y) = x^2 + 2xy$ so:

$$\frac{\partial F_2}{\partial x} = 2x + 2y$$

so $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, F is conservative.

Find the Potential Function

Find potential function $f(x,y)$ for $F(x,y) = (2xy+y^2)i + (x^2+2xy)j$

1) Solve $\frac{\partial f}{\partial x} = F_1(x,y) = 2xy+y^2$

- Integrate with respect to x.

$$f(x,y) = x^2y + xy^2 + g(y)$$

where $g(y)$ is a function of y alone

2) $\frac{\partial f}{\partial y} = F_2(x,y) = x^2+2xy$

Differentiate $f(x,y)$ with respect to y:

$$\frac{\partial f}{\partial y} = x^2+2xy+g'(y)$$

Set $F_2(x,y) = x^2+2xy$

$$g'(y) = 0 \Rightarrow g(y) = C$$

Potential Function: $f(x,y) = x^2y + xy^2 + C$

① Integrate first term with respect to x, then add a $C(y)$. Then integrate second term with respect to y.

② Then put together.

Line Integral Using Fundamental Theorem

$\int_C F\cdot dr$, where $F = \nabla f$, $f(x,y) = x^2y + xy^2$, and C is from $p(1,0)$ to $q(0,2)$

1.) Apply Theorem:

$$\int_C F\cdot dr = f(Q) - f(P)$$

2) Find $f(P)$ and $f(Q)$

$f(P) = (1,0) = 1^2(0) + (1)\cdot 0^2 = 0$

Plug in to $f(x,y)$

$f(Q) = 0^2\cdot 2 + 0\cdot 2^2 = 0$

3) Find $\int_C F\cdot dr = f(Q) - f(P) = 0 - 0 = 0$

Basically: If $F = \nabla f$ meaning if f is a gradient field. The line integral only depends on the start and end points.

Note: When a vector field is path independent, we can define the potential energy, which only depends on the starting/endpoints.

Lecture 26

Gradient Vector Fields

A gradien Vector Field F is derived from the gradient of a scalar potential function $f(x,y,z)$:

$$F = \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

- If F is a gradient vector field, the line integral is Path independent: $\int_C F \cdot dr = f(Q) - f(P)$

  where P and Q are start/end points.

More on Path Independence

Path Independence- line integral only depends on endpoints.

Circular Test: The line integral around any closed curve must vanish:

$$\int_C F \cdot dr = 0$$

Curl Test: For F in $\mathbb{R}^2$:

$$\text{curl } F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

- If curl F = 0, F is a gradient field
- If curl F ≠ 0, F is not gradient field

Ex: Testing Vector Field Conservative

Det: $F(x,y) = 2xyi + xyj$ is a gradient field.

1) Find partials $F_1(x,y) = 2xy$ $\qquad \frac{\partial F_1}{\partial y} = 2x$

$F_2(x,y) = xy$ $\qquad \frac{\partial F_2}{\partial x} = y$

2) Curl: $\text{curl } F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y - 2z$

Since $F \neq 0$. This is not a gradient vector field.

Greens Theorem

Greens theorem helps calculate the area around a closed curv C, to the double integral over the enclosed Region.

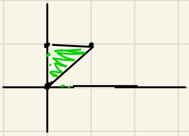
Line integrals: $\int_C P\,dx + Q\,dy$

$$\int_C F\cdot dr = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$\int P\,dx + Q\,dy$$

Ex: Using Greens Theorem

$$\oint_C x\,dx - x^2y^2\,dy$$



$$\int P\,dx - Q\,dy$$

Let $P = x$
Let $Q = x^2y^2$

$$\int \frac{dP}{dx} - \frac{dQ}{dy}$$

$$\int_0^1\int_x^1 (-2xy^2) - 0\,dy\,dx$$

$$= \int_x^1 -2xy^2 = -2x\cdot\frac{y^3}{3}\Big|_x^1$$

$$= -2x\cdot\frac{1}{3} - \left[-2x\cdot\frac{x^3}{3}\right]$$

$$= \frac{-2x}{3} + \frac{2x^4}{3}$$

$$\int_0^1 \frac{-2x}{3} + \frac{2x^4}{3} \qquad \frac{-2x^2}{6} + \frac{2x^5}{15}\Big|_0^1$$

$$= \frac{-1}{15}$$

Example 2

$x = 3\cos t$
$y = 3\sin t$

$$\int P\,dx + Q\,dy$$

$$\iint \frac{dQ}{dx} + \frac{dP}{dy}$$

$$\int 3\cos t + 3\sin t$$

Lecture 27

- A line integral computes work done by a force field along a curve.
- Force Field $F = F_1 i + F_2 j + F_3 k$
- A curve C is parameterized as:

$$r(t) = x(t) i + y(t) j + z(t) k, \quad a \leq t \leq b$$

where $r(a) = P$ start

$r(b) = Q$ end

How to Solve Line Integral Questions

1.) Determine if it's a scalar field or vector field.

2.) Parameterize curve, use direction vector to help parameterize.

3.) Compute dr/ds (dr is the derivative of the parametrized position vector.)

4.) Substitute into line integrals:

$$\int_C F \cdot dr = \int_a^b F(x(t), y(t), z(t)) \cdot \frac{dr(t)}{dt} \cdot dt.$$

5. Determine bounds, from parameter t that defines the curve. Evaluate Integral

---

Example 1

$\int_C F \cdot dr$ where $F = (2x + y)i + (x - y)j$

C is line from P(2,1) to Q(0,3)

1) Parameterize: $\vec{PQ} = (-2, 2)$

= Point + t directionVector

$= (2, 1) + t(-2, 2)$

$x = 2 - 2t$

$y = 1 + 2t$

2) $r(t) = (x(t), y(t))$

$r(t) = ((2 - 2t + 1 + 2t))$

$r'(t) = \langle -2i + 2j \rangle$

3) $F(r(t)) = [2(2-2t) + (1+2t))i + ((2-2t) - (1+2t))j$

$= (5 - 2t)i + (1 - 4t)j$

4) $\int_{0\,C}^{3} F(r(t)) \cdot r'(t)$

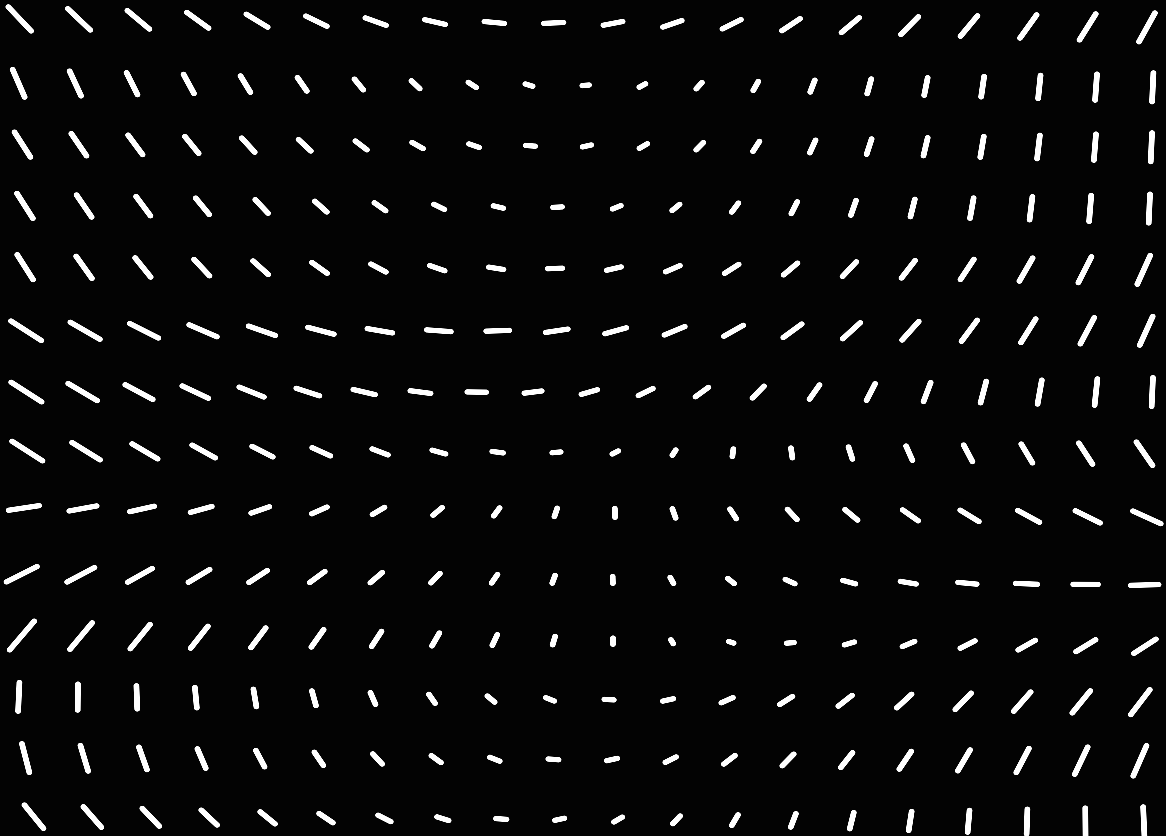
$= \int_{0\,C}^{1} (5-2t) \cdot (-2) + (1-4t) \cdot 2$

$-10 + 4t + 2 - 8t$

$= -8 - 4t$

$= -8t - \frac{4t^2}{2} \Big|_0^1 = -8 - 2 = -10$

Homework Questions



Week 1: Homework Questions

1) $P_1 = (1,2,3)$ $\quad d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

$P2 = (3,2,1)$

$P3 = (1,1,0)$

1) Distance between P1, P2

$d = \sqrt{(1-3)^2 + 0^2 + (3-1)^2}$

$d = \sqrt{8}$

2) Dist between P1, P3

$d = \sqrt{0^2 + 1^2 + 3^2}$

$d = \sqrt{10}$

3) Dist between P2, P3

$d = \sqrt{2^2 + 1^2 + 1^2}$

$d = \sqrt{6}$

∴ The P2, P3 are closest to each other.

2) yz plane, $x=0$

To find the closest point to the yz plane we just see which points x value is closest to 0.

∴ Point B is closest.

$B = (0.9, 0, 3.2) \rightarrow$ Lies on xz plane

↳ Farthest from xy plane.

3) Midpoint Formula: $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$M = \left(\frac{-1+5}{2}, \frac{3+6}{2}, \frac{9+(-3)}{2}\right)$

$M = (2, 4.5, 3)$

4) A Sphere lowest point: $(2,3,-1)$

Centre point: $(2,3,7)$

Radius is $7-(-1) = 8$

∴ Highest pt is $7+8=15$

$z=15$

∴ $(2,3,15)$

5) $y = C$

∴ $y = 3$

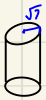
A vertical plane perp to the y axis will always have a constant value.

$(2,3,4) \rightarrow y=3$

6) a) $C = f(d,m) = 40d + 0.15m$

b) $f(5,300) = 40(5) + 0.15(300) = \$245$

Week 2: Homework Questions

1) 

Eq of Circle:
$x^2 + z^2 = r^2$

Because y is remaining the same for all values. It is defined by the circles

$x^2 + z^2 = 7$

2) Radius = 3
centre = $(0, \sqrt{7}, 0)$

General Equation of Sphere: $x^2 + y^2 + z^2 = r^2$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

Sub values

$(x-0)^2 + (y - \sqrt{7})^2 + (z-0)^2 = 3^2$

$x^2 + (y - \sqrt{7})^2 + z^2 = 9$

3) $z = x^2 + y^2$ General Equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$

- vertex (1,3,5)
- parallel to x axis
- opening towards negative x

Standard Formula: $z = x^2 + y^2$

By default paraboloid opens up in the z direction
To make it parallel to x we switch sides.

$z = x^2 + y^2$

has vertex at (0,0,0), opens in positive z direction

In terms of x

To open it along x we do $x = y^2 + z^2$

To open it toward negative x values we get $x = -(y^2 + z^2)$

negative x

Now shifting the vertex

Vertex Shift

$(x-1) = -(y-3)^2 - (z-5)^2$

Final Eq: $x = 1 - (y-3)^2 - (z-5)^2$

4) $f(x,y) = y^3 + xy$. Draw graphs Cross Sections

a) $x = -1$, $x = 0$, $x - 1$

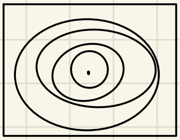
$f(-1, y) = y^3 - y$
$f(0, y) = y^3$
$f(1, y) = y^3 + y$

b) $y = -1$, $y = 0$, $y = 1$

$f(x, -1) = -1 - x$
$f(x, 0) = 0$
$f(x, 1) = 1 + x$

5) Countour Diagrams describe the height or z value of a given x, y.

$f(x,y) = 1 - x^2 - y^2 = C$



radius $\sqrt{1-C}$

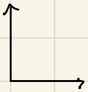
6) $f(x,y) = xy$

$f(x,y) = xy = C$ is the graph of hyperbola $y = \frac{c}{x}$ if $c \neq 0$

The values of C get closer as you move farther away.

7) $f(x,y) = x^2 - y^2 = (x-y)(x+y)$

If $x = 0$, $(-y)(+y) = -y^2$
If $y = 0$, $(x)(-x) = -x^2$



$x - y = 0$ or $x + y = 0$
$y = +x$ or $y = -x$

Perpendicular lines, give 4 angles

8) Linear function: $mx + b$

$z = c + mx + ny$

Plane: $a = (4,0,0)$
$b = (0,3,0)$
$c = (0,0,2)$

$0 = c + 4 \cdot m + 0 \cdot n$
$0 = c + 4m$

$0 = c + 3m + 0$
$0 = c + 3n$

$c = 2$

$c = -4m$ $c = -3m$

$2 = -4m$
$m = -\frac{1}{2}$

$\frac{2}{2} = -\frac{\cancel{3}n}{3} = -\frac{2}{3}$

$z = 2 - \frac{1}{2}x - \frac{2}{3}y$

9) $z = C + mx + ny$ $\quad c, m, n?$

graph intersects the xz plane $(y=0)$
in the line $z = 3x+4$ and yz in $z=y+4$

$z = 3x+4$ $\quad C + mx = 3x+4, \ C=4, m=3$
$z = y+4$ $\quad c + ny = 4 + ny = 4 + y, \ n=1$

$xz: (1,0,1)$ $\quad$ Thus, $z = 3x+y+4$
$yz: (0,1,1)$

10) $f(x) = \dfrac{x^2-y^2}{x^2+y^2}$ does not have
a limit as $(x,y) \to (0,0)$

It does not have a limit due to
asymptotic behaviour. $\lim\limits_{x,y \to 0} f(x,y) = \dfrac{1-m^2}{1+m^2}$.

Continuity means $\lim_{x,y \to 0} f(x,y)$.

Week 3: Homework Questions

1) $\vec{n} = (2,4,-3)$
$P = (1,0,-1)$
$ax + by + cz + d = 0$ → plug into this default formula

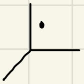
Parallel to plane means Same normal Vector

$2(1) + 4(0) - 3(-1) = 0$
$2 + 0 + 3 + d = 0$
$5 + d = 0$
$d = -5$

EQ: $2x + 4y - 3z - 5 = 0$

2) Finding θ between Vectors
$i + j + k$ and $i - j - k$

$(1,1,1)$
$(1,-1,-1)$



$\cos\theta = \frac{U \cdot V}{|U||V|}$

$\cos\theta = \frac{-1}{3}$

$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$

$\theta = 109.47$

$a \cdot b = |a||b|\cos\theta$

$\theta = \cos^{-1}\left(\frac{a \cdot b}{|a||b|}\right)$

3) Give a unit vector

a) $V = 2i + 3j$
$V = (2,3)$
$U = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$

b) $(2,3) \cdot (x,y) = 0$
$2x + 3y = 0$
Guessing: $x = -3, y = 2$
$2(-3) + 3(2) = 0$
$-6 + 6 = 0$
$0 = 0$

$(-3,2) \rightarrow U = \left(\frac{-3}{\sqrt{13}}; \frac{2}{\sqrt{13}}\right)$

4) Plane: $5x - y + 7z = 21$

a) $5x = 21$
$\frac{21}{5} = x$ → $\left(\frac{21}{5}, 0, 0\right)$

b) $-y = 21$
$y = -21$
$(0, -21, 0)$

$7z = 21$
$z = 3$
$(0,0,3)$

c) $(5,-1,7) \cdot (x,y,z) = 0$
$\vec{P} = (-1,-5,0)$
$5x - 1y + 7z = 0$
$5(-1) - 1(-5) + 7(0) = 0$

d) $\vec{n} = (5,-1,7)$

5) $a = (1, -3, -1), b = (1, 1, 2), c = (-2, -1, 1), d = (-1, -1, 1)$

a) $a \cdot c = 0?$
$(1,-3,-1) \cdot (-2,-1,1) = 0$
$0 = 0$

$b \cdot d = 0?$
$(1,1,2) \cdot (-1,-1,1) = 0$
$-1 - 1 + 2 = 0$
$0 = 0$

b) None of the pairs are scalable to give you the same output.

c) Less than $\frac{\pi}{2}$, means the dot product is positive

$c \cdot d = 4 \quad \therefore c, d$
$a \cdot d = 1 \quad \therefore a, d$

d) More than $\frac{\pi}{2}$, means the dot product is negative.

$a \cdot b = -4 \qquad b \cdot c = -1$

6) $\vec{a} = (3, 2, -6)$
$\vec{a} = a_{paralel} + a_{perp}$
$\vec{d} = (2, -4, 1)$

To find a parallel, the projection of a on to d gives the parallel components

$a_{parallel} = \frac{a \cdot d}{d \cdot d} d$

$= \frac{(3,2,-6) \cdot (2,-4,1)}{(2,-4,1) \cdot (2,-4,1)} (2,-4,1)$

$= \frac{-8}{21}(2,-4,1)$

$= \left(\frac{-16}{21}, \frac{32}{21}, \frac{-8}{21}\right)$

$\vec{a} - a_{parallel} = a_{perp}$

$(3,2,-6) - \left(\frac{-16}{21}, \frac{32}{21}, \frac{-8}{21}\right)$

$= \left(\frac{79}{21}, \frac{10}{21}, \frac{-118}{21}\right)$

$\vec{a} = \left(\frac{79}{21}, \frac{10}{21}, \frac{-118}{21}\right) + \left(\frac{-16}{21}, \frac{32}{21}, \frac{-8}{21}\right)$

7) $a = (1,0,0)$
$b = (0,1,0)$
$c = (0,0,1)$

$\vec{ab} = (-1, 1, 0)$
$\vec{bc} = (0, -1, 1)$

$\vec{ab} \times \vec{bc}$

$\cancel{-1}\ \ 1\ \ 0\ \ -1\ \ 1\ \ \cancel{0}$
$\cancel{0}\ \ -1\ \ 1\ \ 0\ \ -1\ \ \cancel{1}$

$\vec{n} = (1,1,1)$

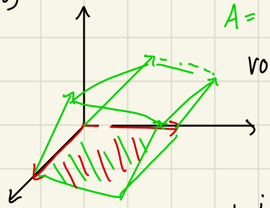
$ax + by + cz + d = 0$
$1 + 0 + 0 + d = 0$
$d = -1$

$\therefore$ EQ: $x + y + z - 1 = 0$

$\cos\theta = \frac{U \cdot V}{|U||V|}$

$a_{parallel} = \frac{a \cdot d}{d \cdot d} \vec{d}$

8)



$A = \|v \times w\|$

$\text{Volume} = \underset{\text{height}}{u} \cdot \underset{\text{area}}{(v \times w)}$

$$u \cdot (v \times w) = \left| \begin{matrix} i & j & k \\ 5 & 4 & 3 \\ 1 & 1 & 1 \end{matrix} \right| \qquad \det = i - 2j + k$$

$$A = \left| \begin{matrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 3 & 4 & 5 \end{matrix} \right| \qquad \det(A) = 0$$

A parallelepiped may have volume 0, iff a belongs to the plane spanned by vectors b and c.

a) $\vec{v} = (x, y, z)$

Intersection of the planes:

$2x - 3y + 5z = 2$
$4x + y - 3z = 7$

$$\begin{matrix} \cancel{2} & -3 & 5 & 2 & -3 & \cancel{5} \\ \cancel{4} & 1 & -3 & 4 & 1 & \cancel{-3} \end{matrix}$$

$\vec{n_1} \times \vec{n_2} = (4, 26, 14)$

$\vec{n}$ is perpendicular to every vector lying on the plane.

The line of intersection lies in both planes, meaning the line of intersection must be perpendicular to both planes.

10) Perpendicular to 2 planes

$x - y + z = 5$
$2x + y - 2z = 7$

$$\begin{matrix} \cancel{1} & -1 & 1 & 1 & -1 & \cancel{1} \\ \cancel{2} & 1 & -2 & 2 & 1 & \cancel{-2} \end{matrix}$$

2+2

1+2 $\vec{M} = (1, 4, 3)$

$ax + by + cz + d = 0$

$1(0) + 4(0) + (0)(3) =$

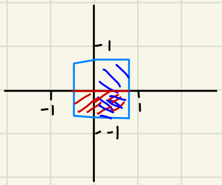
$0 = 0$

$d = 0$

$\therefore$ EQ: $x + 4y + 3z = 0$

Week 4: Homework Questions

1)



a) $\int_D dxdy$ The function $f(x,y)=1$, is positive everywhere. Constant positive function.

b) $\int_R 5x\,dxdy$ The function would be $\frac{5x^2}{2}$, if x is neg, the integral would still be positive. And in R, $x >$

c) $\int_B 5x\,dxdy$ Odd function in x, meaning $f(x,y) = -f(-x,y)$

Because B includes both positive and negative values of x and is symmetric about y axis, they will cancel out. Integral $=0$

d) $\int_D (y^3+y^5)\,dxdy$ D is symmetric about the x axis/y axis. y^3 and y^5 are odd functions of y. This means integral 0.

e) $\int_B (y^3+y^5)\,dxdy$ In B, $y \le 0$, which means it is less than 0 for all y^3+y^5. Negative

f) $\int_D y-y^3\,dxdy$. D is symetric to y and they are odd functions

g) $\int_B y-y^3\,dxdy$ $y-y^3$ is always negative in B and $|y^3| < |y|$
$\therefore$ Negative

Note:

- If the region is positive or negative across the entire region, the integral will be positive or negative.
- Odd: $f(-x) = -f(x)$

If the region is symmetric around y axis and $f(x,y)$ is odd in x. Integral $=0$

If the region is symmetric around x axis and $f(x,y)$ is odd in y. Integral $=0$

2) f and g are continous on a region R.

$$\int_R f \cdot g \, dA = \int_R f \, dA \cdot \int_R g \, dA$$

False, Counter example, if R is a rectangle with area 2 and take $f(x,y) = g(x,y) = 1$ Then $\int_R fg \, dA$ is going to mean Area(R) = 2, but $\int_R f \, dA \cdot \int_R g \, dA$

$= \text{Area}(R) \cdot \text{Area}(R) = 4$

3a) $$\int_0^1 \int_0^1 y e^{xy} \, dx \, dy$$

$$\int_0^1 y e^{xy} \, dx = \cancel{y} \cdot \frac{e^{xy}}{\cancel{y}} \Big|_{x=0}^{x=1}$$

$= (e^y - 1) dy$

$= \int_0^1 (e^y - 1) dy = e^y - y \Big|_0^1$

$= (e - 1) - (1 - 0)$

$= e - 2$

b) $$\int_0^{\frac{\pi}{2}} \int_0^{\sin x} x \, dy \, dx$$

$$\int_0^{\sin x} x \, dy = xy \Big|_{y=0}^{y = \sin x}$$

$= x \cdot \sin x - x \cdot 0$

$= x \cdot \sin x$

$\int u \, dv = uv - \int v \, du$

$$\int_0^{\frac{\pi}{2}} x \cdot \sin x \, dx$$

$u = x$

$\frac{du}{dx} = \frac{1}{1}$

$dx = du$

$dv = \sin x \, dx$

$v = -\cos x$

$= -x \cos x + \int_0^{\frac{\pi}{2}} + \cos x \, dx$

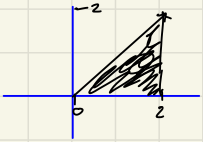
$= -\left(\frac{\pi}{2}\right) \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)$

$= 0 + 1$

$= 1$

Week 5: Homework Questions

1) $\int_0^2 \int_0^x e^{x^2} dy\, dx$



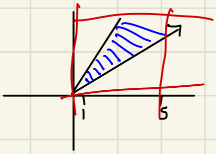
$\int_0^x e^{x^2} dy = e^{x^2} \cdot y \Big|_0^x = e^{x^2} \cdot x$

$\int_0^2 e^{x^2} \cdot x \Big|_0^2 =$

$u = x^2$
$du = 2x\, dx$
$dx = \frac{1}{2} du$

$= \frac{1}{2} \int_0^4 e^u du = \frac{1}{2} e^u \Big|_0^4 = \frac{1}{2}(e^4 - 1)$

2.) $\int_1^5 \int_x^{2x} \sin x \, dy\, dx$



$\int_x^{2x} \sin x \, dy = \sin x \cdot y \Big|_x^{2x}$

$= 2x \cdot \sin x - x \sin x \rightarrow (\sin x(2x - x) dx)$

$= \int_1^5 x \sin(x) dx =$

By Parts: $\int u\, dv = uv - \int v\, du$

$= x(-\cos x) - \int -\cos x \, dx$

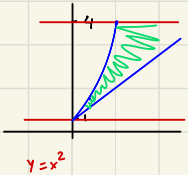
$= -\cos x \cdot x + \sin x \Big|_1^5$

$= -5\cos 5 + \cos 1 + \sin 5 - \sin 1$

$u = x$
$du = 1$
$dx = du$
$dv = \sin x$
$v = -\cos x$

3) $\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 dx\, dy$

y is bounded by 1, 4
x is bounded by y, $\sqrt{y}$



$\int_{\sqrt{y}}^{y} x^2 y^3 dx = \frac{x^3 y^3}{3} \Big|_{x=\sqrt{y}}^{x=y}$

$= \left[\frac{y^6}{3} - \frac{\sqrt{y}^3 \cdot y^3}{3}\right]$

$= \frac{y^9}{3} - \frac{y^4 \cdot y^{\frac{1}{2}}}{3}$ or $\frac{y^2}{3}$

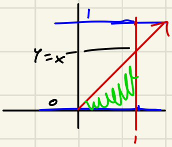
$(\sqrt{y})^3 = y^{\frac{3}{2}}$

$\int_1^4 \frac{y^9}{3} - \frac{y^2}{3} = \frac{y^{10}}{3 \cdot 10} - \frac{y^3}{3 \cdot 3} \Big|_1^4$

$= \frac{524288}{15} - \frac{64}{9} - \frac{1}{30} + \frac{1}{9}$

4) $\int_0^1 \int_y^1 e^{x^2} dx\, dy$

x is bounded by 1 and y
y is bounded by 0, 1



$\int_0^x e^{x^2} dy = e^{x^2} \cdot y \Big|_{y=0}^{y=x}$

$= x \cdot e^{x^2}$

$= \int_0^1 x \cdot e^{x^2}$

$u = x^2$
$du = 2x$
$dx = \frac{1}{2} x$

$= \int_0^1 e^u du = \frac{1}{2} e^u \Big|_0^1 = \frac{e-1}{2}$

5) $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\, dx\, dy$

y is bounded from 0 - 1
x is bounded from $\sqrt{y}$ - 1
$x = \sqrt{y}$
$y = x^2$

$\int_0^1 \int_0^{x^2} \sqrt{2+x^3} \quad dy\, dx$

$\int_0^{x^2} \sqrt{2+x^3} = \sqrt{2+x^3} \cdot y \Big|_0^{x^2}$

$= \int_0^1 x^2 (2+x^3)^{\frac{1}{2}} dx$

$u = 2 + x^3$
$du = 3x^2$
$dx = \frac{1}{3x^2} du$

$= \int_2^3 \sqrt{u} \cdot \frac{du}{3}$

$= \frac{1}{3} \int_2^3 (u)^{\frac{1}{2}} du = \frac{u^{3/2}}{\frac{3}{2}} = \frac{2}{3} u^{3/2}$

$= \frac{1}{3} \cdot \frac{2}{3} \left(u^{\frac{3}{2}}\right) \Big|_2^3$

$= \frac{2}{9}\left(3\sqrt{3} - 2\sqrt{3}\right)$

6) $\int_0^3 \int_{y^2}^9 y \sin(x^2)\, dx \cdot dy$

y is bounded by 0-3
x is bounded by y^2- 9

$x = y^2$
$y = \sqrt{x}$

$\int_0^9 \int_0^{\sqrt{x}} y \sin(x^2) dy\, dx$

$\int_0^{\sqrt{x}} y \sin(x^2) dy = \frac{y^2}{2} \cdot \sin(x^2) \Big|_0^{\sqrt{x}}$

$= x \cdot \sin(x^2)$

$\int_0^9 \frac{x}{2} \cdot \sin(x^2)\, dx$

$\int u\, dv = uv - \int v\, du$

$u = x$
$du = 1$
$dx = du$
$dv = \sin x^2$
$v = -\cos x^2$

$= \frac{1}{2}\, x \cdot (-\cos x^2) - \int -\cos x^2 \, dx$

$= \frac{1}{2}\, -x \cdot \cos x^2 + \sin x^2 \Big|_0^9$

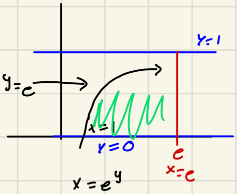
$= 0.056$

⑦ $\int_0^1 \int_{e^y}^{e} \frac{x}{\ln x}\, dx\, dy$

y is bounded by 0-1

x is bounded by e^y - e

$\int_1^e \int_0^{\ln x} \frac{x}{\ln x}\, dy\, dx$



$\int_0^{\ln x} \frac{x}{\ln x}\, dy = \frac{x}{\ln x} \cdot y \Big|_0^{\ln x}$

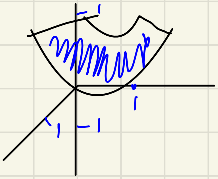
$x = e^y$
$y = \ln x$

$\int_1^e x\, dx = \frac{x^2}{2}\Big|_1^e = \frac{1}{2}(e^2-1)$

8) a) Below: → $z = x^2+y^2$

Above: → x, Y plane or $z=0$

$\int_{-1}^{1}\int_{-1}^{1} x^2+y^2\, dx\, dy$



$\int_{-1}^{1} x^2+y^2\, dx = \frac{x^3}{3} + y^2 \cdot x\Big|_{-1}^{1}$

$= \frac{1}{3} + y^2 + \frac{1}{3} + y^2$

$= \int_{-1}^{1} \frac{2}{3} + 2y^2\, dy = \frac{2}{3}y + \frac{2y^3}{3}\Big|_{-1}^{1}$

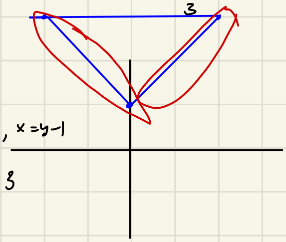
$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} - \frac{2}{3} = \frac{8}{3}$

b) The region above the surface and below $z=2$, together with volume of region 1, gives us. $V = 2^3 \to 8$

The volume under that is $V = 8 - \frac{8}{3}$

$V = \frac{16}{3}$

9) $\iint_R (2x^2+y)\, dA$



x is bounded by $x=1-y$, $x=y-1$

Y is bounded by 1 and 3

$\int_1^3 \int_{1-y}^{y-1} (2x^2+y)\, dx\, dy$

$= \int_{1-y}^{y-1} (2x^2+y)\, dx = \frac{2x^3}{3} + xy\Big|_{1-y}^{y-1}$

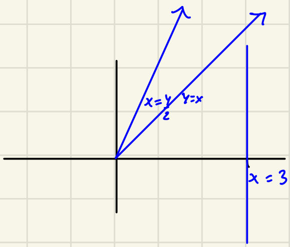
$= \left[\frac{2(y-1)^3}{3} + (y-1)(y)\right] - \left[\frac{2(1-y)^3}{3} + (1-y)(y)\right]$

$2\int_{-1}^{3} \frac{2}{3}y^3 - y^2 + y - \frac{2}{3}\, dy = \frac{2y^4}{3\cdot 4} - \frac{y^3}{3} + \frac{y^2}{2} - \frac{2}{3}y\Big|_{-1}^{3}$

$= \frac{44}{3}$

10) $f(x,y) = x^2 e^{x^2}$

and R is triangle bounded by $X=3$, $x=\frac{y}{2}$, $Y=x$



$x = \frac{y}{2}$

$Y = 2X$

First way: x bounded by $x=0$ and $x=3$
y bounded by x and 2x

$\int_0^3 \int_x^{2x} x^2 e^{x^2}\, dy\, dx$

Second way:

$\int_0^3 \int_{y/2}^{y} x^2 e^{x^2}\, dx\, dy + \int_3^6 \int_{y/2}^{3} x^2 e^{x^2}\, dx\, dy$

$\int_0^3 \int_x^{2x} x^2 e^{x^2}\, dy\, dx$

$\int_x^{2x} x^2 e^{x^2}\, dy = x^2 e^{x^2} \cdot y\Big|_x^{2x}$

$= x^2 e^{x^2} \cdot 2x - x^2 e^{x^2} \cdot x$

* $\int_0^3 x^3 \cdot e^{x^2}\, dx$

BP: $\int u\, dv = uv - \int v\, du$

$u = e^{x^2}$
$du = 2x \cdot e^{x^2}$
$dv = x^3$
$v = \frac{1}{2}$

Week 6: Homework Questions

1) $f(x) = 2x + y + z = 4$

$z = 4 - 2x - y$

The plane intersects the xy plane at $2x + y = 4$

This will be what bounds the region in the first quadrant

$2x + y = 4$ intersects x at $2x = 4$, $x = 2, 0$

y at $0, 4$

$$\int_0^2 \int_0^{4-2x} 4 - 2x - y \, dy \, dx$$

$$= \int_0^{4-2x} 4 - 2x - y \, dy = 4y - 2xy - \frac{y^2}{2} \Big|_0^{4-2x}$$

$$= 4(4-2x) - 2x(4-2x) - \frac{(4-2x)(4-2x)}{2}$$

$$= \frac{1}{2} \int_0^2 (4-2x)^2 \, dx = 16 - 16x + 4x^2$$

$$= 16x - \frac{16x^2}{2} + \frac{4x^3}{3} \Big|_0^2$$

$$= \frac{1}{2}\left(-0 + \frac{4^3}{6} \right)$$

$$= \frac{16}{3}$$

2) $f(x,y,z) = e^{-x-y-z}$

corners at $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$

$$\int_W f \, dV = \int_0^a \int_0^b \int_0^c e^{-x-y-z} \, dz \, dy \, dx$$

$$\int_0^c e^{-x} \cdot e^{-y} \cdot e^{-z} \, dz = e^{-x} \cdot e^{-y} (-e^{-z}) \Big|_0^c$$

$$= (1 - e^{-c}) e^{-x} \cdot e^{-y}$$

$$\int_0^b (1 - e^{-c}) e^{-x} \cdot e^{-y} \, dy = -e^{-y} \Big|_0^b$$

$$= (1 - e^{-c})(1 - e^{-b}) e^{-x} \int_0^c e^{-x} = 1 - e^{-x} \Big|_0^c$$

$$= \int_W f \, dV = (1 - e^{-a})(1 - e^{-b})(1 - e^{-c})$$

Note: You can factor out variables if they are independent of each other.

3) Bounded by xy plane: $x + y = 2 \rightarrow x = 2$, $x = 2 - y$

when x and $y = 2 \rightarrow y = 2$

$y = 0$

Bounded by xz plane: $x + z = 2$

Bounded by yz plane: $y + z = 2$

$z = 2 - x$

$z = 0$

$z = 2 - x - y$

1.) x axis $= (2,0,0)$

y axis $= (0,2,0)$

z axis $= (0,0,2)$

$$\int_0^2 \int_0^{2-x} \int_0^{2-x-y} ??? \, dz \, dy \, dx$$

$$\int_0^{2-x} (2 - x - y) \, dy \, dx = \int_0^2 2y - xy - \frac{y^2}{2} \Big|_0^{2-x} dx$$

$$= \int_0^2 \frac{x^2}{2} - 2x + 2 \, dx = \frac{x^3}{6} - x^2 + 2x \Big|_0^2 = \frac{4}{3}$$

4) $\int_R \sin(x^2+y^2)\,dA$, where R is the disk of radius 2 centered at origin

$$\int_0^{2\pi}\int_0^{2}(\sin r^2) \qquad dA = r\,dr\,d\theta$$

$$\int_0^2 \sin r^2 \cdot r\,dr$$

$u = r^2$
$du = 2r\,dr$
$\frac{du}{2} = r\,dr$

$$\int_0^4 \sin u \cdot \frac{du}{2}$$

$$= \frac{1}{2}\int_0^4 \sin u\,du$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$= \frac{1}{2} -\cos(u)\Big|_0^4 = -\frac{1}{2}\cos(4) - \cos(0)$$

$$= \frac{1}{2}(1-\cos(4)) \cdot 2\pi$$

$$= \pi(1-\cos(4))$$

5) $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx \rightarrow r\,dr\,d\theta$

From the limits of integration we see this is the left half of the disk $x^2+y^2 \leq 1$.

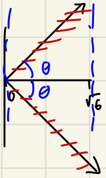
$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^1 r\cos\theta\; r\,dr\,d\theta$$

$$\int_0^1 r \cdot r\,dr = \frac{r^3}{3}\Big|_0^1$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{3}\cos\theta$$

$$= \frac{1}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta = \sin\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{1}{3}(-1-1) = \frac{2}{3}$$

6) $\int_0^{\sqrt{6}}\int_{-x}^{x} dy\,dx \rightarrow$ Polar Coordinate



x is bounded by 0, $\sqrt{6}$
y is bounded by $-x$, x

$r, \theta = ?$

$x = r\cos\theta$
$y = r\sin\theta$

Seeing the lines $y = x$ and $y = -x$ we see $\theta = \frac{\pi}{4}$ and $\frac{-\pi}{4}$.

we know $0 \leq r\cos\theta \leq \sqrt{6}$
r from 0 to $\frac{\sqrt{6}}{\cos\theta}$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\frac{\sqrt{6}}{\cos\theta}} r\,dr\,d\theta$$

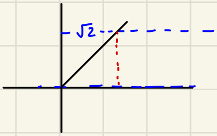
$$\int_0^{\frac{\sqrt{6}}{\cos\theta}} r\,dr = \frac{r^2}{2}\Big|_0^{\frac{\sqrt{6}}{\cos\theta}}$$

$$= \frac{6}{2\cos^2\theta} = 3\tan\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 3 - -3 = 6$$

7) $\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xy\,dx\,dy$ to polar coordinate.

x is bounded by y and $\sqrt{4-y^2}$
y is bounded by 0 and $\sqrt{2}$



$y = x$

$x = \sqrt{4-y^2}$
$x^2 = 4-y^2$
$x^2-4 = -y^2$
$-\sqrt{x^2-4} = y$

$x = r\cos\theta$
$y = r\sin\theta$

$r = x^2+y^2$

$r\cos\theta = \sqrt{4-y^2}$

$$\int_0^{\frac{\pi}{4}}\int_0^{2} r\cos\theta \cdot r\sin\theta \cdot r \cdot dr \cdot d\theta$$

$$\int_0^2 r^3 \cos\theta\sin\theta \quad dr$$

$$\int_0^2 r^3\,dr = \frac{r^4}{4}\Big|_0^2 = 4$$

$$\int_0^{\frac{\pi}{4}} \cos\theta\sin\theta\,d\theta =$$

$\cos\theta\sin\theta = \frac{1}{2}\sin(2\theta)$

$u = 2\theta$
$du = 2$
$d\theta = \frac{du}{2}$

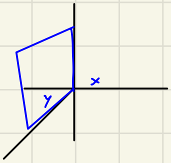
$$= \frac{1}{2}\int \sin(2\theta)\,d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(u)\,du = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin(u)\,du$$

$$= \frac{1}{4}\left[-\cos(u)\right]_0^{\pi/2} = \frac{1}{4}$$

$$= 4 \cdot \frac{1}{4} = 1$$

8) a) Cylindircal : (r, θ, z)
Spherizal : (ρ, ϕ, θ)



$y = x$ creates 45° with Positive x axis
∴ Eq = $\theta = 45°$

b) $x^2+y^2+z^2=1$ in Cylindrical Coordinates

radius 1, centered at origin

$$r^2+z^2=1$$

For Top half $z \geq 0$

$$z = \sqrt{1-r^2}$$

8c) The cone $\sqrt{x^2+y^2}$ in Cylindrical Coordinates

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$z = r$

8d) The cone $\sqrt{x^2+y^2}$ in Spherical coordinates

$z = \rho\cos\phi$ and $r = \rho\sin\phi$

So $z = \sqrt{x^2+y^2}$ becomes

$$\rho\cos\phi = \rho\sin\phi$$

$$\cos\phi = \sin\phi$$

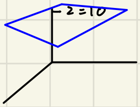
$\tan\phi = 1$ so:

$$\phi = \frac{\pi}{4}$$

8e) The Plane $z=10$ in Spherical coordinates

Spherical: (ρ, ϕ, θ)

Plane $z=10$



$z = \rho\cos\phi$

$\rho\cos\phi = 10$

$\rho = \frac{10}{\cos\phi}$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

The Equation in Spherical Coordinates is $\rho = \frac{10}{\cos\phi}$

8f) The plane $z=4$ in Spherical coordinates

Spherical: (ρ, ϕ, θ)

$z = 4$

$z = \rho\cos\phi$

$4 = \rho\cos\phi$

$\rho = \frac{4}{\cos\phi}$

* solving

9) Evaluate Triple integral of $f(x,y,z) = \sin(x^2+y^2)$ in cylindrical coordinates over cylinder w height 4. and $r=1$ centered at $z=-1$.

Form: $f(r, \theta, z)$

$x = r\cos\theta$

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

z is bounded by -1 and

r is bounded by 0 and 1

θ is 0 to 2π

height of 4 Starting at -1 Means -1 to 3

$$\int_{-1}^{3}\int_{0}^{2\pi}\int_{0}^{1} (\sin r^2) r \cdot dr \cdot d\theta\, dz$$

$$= \int_0^1 (\sin r^2) r \cdot dr = \int_{-1}^{3}\int_{0}^{2\pi} \frac{-1}{2}\cos r^2 \Big|_0^1 d\theta\, dz =$$

$$= -\frac{1}{2}\int_0^{2\pi} (\cos 1 - 1)\, d\theta = -(\cos 1 - 1)$$

$$= \int_{-1}^{3} dz = -4\pi(\cos 1 - 1) = 4\pi(1-\cos 1)$$

10) $f(x,y,z) = x^2+y^2+z^2$

where $0 \le r \le 4$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$, $-1 \le z \le 1$

$$\int_{-1}^{1}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{0}^{4} r^2+z^2\, r\,dr\, d\theta\, dz$$

$$\int_0^4 r^2+z^2\, r\,dr = \frac{r^4}{4}\Big|_0^4 + \frac{z^2 \cdot r^2}{2}\Big|_0^4$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 64+8z^2\, d\theta\, dz = (64+8z^2)\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$$

$$\int_{-1}^{1} \frac{\pi}{2}(64+8z^2)\, dz$$

$$\frac{\pi}{2}\int_{-1}^{1} dz = 8 \cdot \frac{z^3}{3}\Big|_{-1}^{1} = 8 \cdot \frac{2}{3} = \frac{16}{3}$$

$$= \frac{\pi}{2}\left(128 + \frac{16}{3}\right) = \frac{200\pi}{3}$$

Week 7: Homework Questions

3) a) The cone $z = \sqrt{x^2+y^2}$

Spherical coordinates: (ρ, ϕ, θ)

$z = \rho\cos\phi \qquad r = \rho\sin\phi$

$r = \sqrt{x^2+y^2} \quad z = r$

$\rho\cos\phi = \rho\sin\phi$

$\cos\phi = \sin\phi$

$\tan\phi = 1$ so:

$\phi = \frac{\pi}{4}$

b) The plane $z = 10$

$z = \rho\cos\phi$

$10 = \rho\cos\phi$

$\rho = \frac{\cos\theta}{10}$

4) $F(\rho, \theta, \phi) = \sin\phi$, over the region $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{4}$, $1 \le \rho \le 2$.

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_1^{2} \sin\phi \; \rho^2 \sin\theta \; d\rho \, d\phi \, d\theta$$

$$\sin^2\phi = \frac{1-\cos 2\phi}{2}$$

$$\frac{\rho^3}{3}\Big|_1^2 = \frac{7}{3} \rightarrow \int_0^{\frac{\pi}{4}} \frac{7}{3}\sin^2\phi \, d\phi \, d\theta$$

$$= \frac{7}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\phi}{2} \, d\phi \, d\theta$$

$$= \frac{7}{6}\int_0^{2\pi}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_0^{\frac{\pi}{4}} = \frac{7}{6}\int_0^{2\pi}\left(\frac{\pi}{4} - \frac{1}{2}\right)d\theta$$

$$= \frac{7\pi(\pi-2)}{12}$$

5) $f(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$

radius of 5, centered at origin

Should take form: $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$\rho = r\cos\phi\sin\phi$

$\rho^2 = x^2+y^2+z^2 \qquad f(x,y,z) = \frac{1}{\rho}$

where ρ reperesents the distance from origin

$$\underbrace{\iiint_W \frac{1}{\rho}\cdot\rho^2}_{=\rho} \sin\phi \, d\rho \, d\theta \, d\phi$$

ρ goes from 0 to 5

ϕ goes from $\frac{\pi}{2}$ to π

θ goes from 0 to 2π

$$\int_0^{5}\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \rho\sin\phi \, d\phi \, d\theta \, d\rho$$

$$\int_{\frac{\pi}{2}}^{\pi} \rho\sin\phi \, d\phi = \rho\int \sin\phi \, d\theta = \rho \; -\cos\theta\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 1 \rightarrow \int_0^{2\pi} 1 \, d\theta = 2\pi$$

$$= \int_0^{5} 2\pi \, d\rho \rightarrow 2\pi\int_0^{5} \rho \, d\rho = 2\pi\cdot\frac{25}{2} = 25\pi$$

6) $\int_W dV$ For $\frac{1}{8}$ of the ball of radius 1 in the octant $x>0, y>0, z<0$.

a) $(x,y,z) \rightarrow$ Sphere can only reach as far a $x=1$

$f(x,y,z) = x^2+y^2+z^2=1$

y is bounded by $y^2=1-z^2-x^2$

$y=\sqrt{1-x^2}$

z is bounded by $z=-\sqrt{1-y^2-x^2}$

Basically, choose x to be bounded by the radius, then isolate for y in the eq, since it relies on x, then isolate for z which is a f(x) of x and y.

CC →
$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{-\sqrt{1-y^2-x^2}}^{0} x^2+y^2+z^2 \; dz\,dy\,dx$$

b) Cylindrical Coordinates: (r,θ,z)

r is given as $x^2+y^2+z^2=1$

$r=1$

Since the sphere is in the first octant and z must be negative

θ is bounded by 0 and $\frac{\pi}{2}$ → Because $x>0, y>0$ in the first octant

z should still be bounded by $0, -\sqrt{1-x^2-y^2}$

$$\int_0^1\int_0^{\frac{\pi}{2}}\int_{-\sqrt{1-r^2}}^{0} r\,dz\,d\theta\,dr$$

c) For Spherical coordinate: (ρ,ϕ,θ)

$x=\rho\cos\phi\sin\phi$ $\quad r=1$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

ρ will be bounded from 0 to 1 as its the distance from the origin.

ϕ will be bounded by $\frac{\pi}{2}$ and π

θ will be 0 to $\frac{\pi}{2}$, since first quadrant

$$\int_0^1\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi} \rho^2\sin\phi\,d\phi\,d\theta\,d\rho$$

Tips

Use Direct limits for Cartesian x, y, z coordinates.

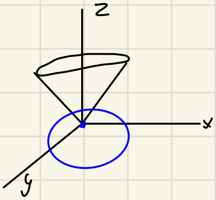
Cylindrical: Convert circular parts to r and θ and set limits for z

Spherical. Use ρ for radial distance φ angle from z axis and θ for rotation around z axis.

radius=1 means r goes 0-1 and ρ- 0-1

For θ look for quadrants

7) cone $z = \sqrt{x^2+y^2}$
topped by sphere of radius
1 centered at origin



a) Cartesian: (x, y, z)

x is bounded by the intersection of the cone and sphere.

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 1$$

$$x^2+y^2+x^2+y^2 = 1 \Rightarrow 2(x^2+y^2)$$

$$= 1 \Rightarrow x^2+y^2 = \frac{1}{2}$$

The cone and sphere intersect in a circle of radius $\frac{1}{\sqrt{2}}$ in x-y plane

$$\text{so } -\frac{1}{\sqrt{2}}\, x \leq \frac{1}{\sqrt{2}}$$

Y is bounded by $-\sqrt{\frac{1}{2}-x^2} \leq y \leq \sqrt{\frac{1}{2}-x^2}$

Z is bounded by upper surface $= \sqrt{1-x^2-y^2}$
and lower surface $=$ $z = \sqrt{x^2+y^2}$

$$\text{so } \sqrt{x^2+y^2} \leq Z \leq \sqrt{1-x^2-y^2}$$

$$\int_{\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$

b) Cylindrical coordinates: (r, θ, z)

r is the radius, in this case r goes 0 to 1

θ is the angle from z axis, $0 - 2\pi$

z is the height at a given point

$z = r$ → bounded by $0, \sqrt{r}$

The sphere becomes $r^2+z^2 = 1$ or $z = \sqrt{1-r^2}$

Z goes from r to $\sqrt{1-r^2}$

$$\int_0^1 \int_0^{2\pi} \int_r^{\sqrt{1-r^2}} r\, dz\, d\theta\, dr$$

c) Spherical coordinates (ρ, ϕ, θ)

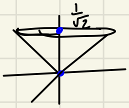
$$\int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

ρ is going from 0 to 1 as its distance of origin to radius

ϕ is the angle formed with Z axis, since the cone angle $\phi = \frac{\pi}{4}$, 0

θ is 0 to 2π because we cover the full circles of sphere

8) $\int_W dV$, for the cone $z = \sqrt{x^2+y^2}$



$$\int_{-\frac{1}{\sqrt{2}}}^{\sqrt{\frac{1}{2}}} \int_{\sqrt{\frac{1}{2}-x^2}}^{-\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\frac{1}{\sqrt{2}}}$$

Cartesian Coordinates: (x,y,z)

z is bounded by $z = \frac{1}{\sqrt{2}}$ and $\sqrt{x^2+y^2}$

y is bounded by $\frac{1}{\sqrt{2}} = \sqrt{x^2+y^2} = x^2+y^2 = \frac{1}{2} \longrightarrow -\sqrt{\frac{1}{2}-x^2}$ to $\sqrt{\frac{1}{2}-x^2}$

x is bounded by Looking at cross section of circle with radius $\sqrt{\frac{1}{2}}$.

Cylindrical coordinates: (r, θ, z)

z is the same at $z = \frac{1}{\sqrt{2}}$ and r

θ goes from 0 to 2π cause it covers the whole circle

r goes from $z = r$, so 0 to $\frac{1}{\sqrt{2}}$

$$\int_0^{\frac{1}{\sqrt{2}}} \int_0^{2\pi} \int_r^{\frac{1}{\sqrt{2}}} r\,dr\,d\theta\,dz$$

Spherical Coordinates: (ρ, ϕ, θ)

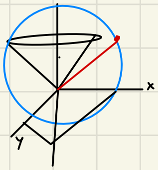
ρ is the distance from origin to point, $\frac{1}{\sqrt{2}} = \rho\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}\cdot\frac{1}{\sqrt{2}}} = 1$

ϕ is the angle formed with z axis which should be $\frac{\pi}{4}$

θ is gonna be $0 - 2\pi$

$$\int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^2 \sin\phi\,d\phi\,d\theta\,d\rho$$

9)



Volume of the solid can be reperesented as a triple integral in Spherical coordinate.

Spherical Coordinate: $x^2+y^2+z^2=8$

ρ is the distance from 0 to point so 0 to $\sqrt{8}$

ϕ is going to $\frac{\pi}{2}$ to $\frac{\pi}{4}$

θ is going to 0 to 2π

$$\int_0^{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{8}} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^{\sqrt{8}} \frac{\rho^3}{3}\Big|_0^{\sqrt{8}} = \frac{16\sqrt{2}}{3}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi\,d\phi = -\cos\phi\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 0 + \frac{\sqrt{2}}{2}$$

$$\int_0^{2\pi} = \frac{16\sqrt{2}}{3} \cdot \frac{\sqrt{2}}{2} \cdot 2\pi = \frac{32\pi}{3}$$

10) Volume of the region inside $z = \sqrt{3(x^2+y^2)}$ below the plane $z=1$

In cylindrical coordinates, cone has eauation $z = \sqrt{3}\cdot r$. When $z=1$, we have $r = \frac{1}{\sqrt{3}}$. so

$$\text{Volume} = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{3}}} \int_{\sqrt{3}\cdot r}^{1} r\,dz\,dr\,d\theta$$

$$= 2\pi \int_{\sqrt{3}\cdot r}^{1} r\,dz = 2\pi \int_0^{\frac{1}{\sqrt{3}}} (1-\sqrt{3}r)\,r\,dr$$

$$= 2\pi\left(\frac{r^2}{2} - \frac{\sqrt{3}}{3}r^3\right)\Big|_0^{1/\sqrt{3}} = 2\pi\left(\frac{1}{6} - \frac{1}{9}\right)$$

$$= \frac{\pi}{9}$$

Homework # 8

1a) Partial Derivatives

$f(x) = x \cdot e^{\sqrt{xy}}$

$f(x) = x \cdot e^{(xy)^{\frac{1}{2}}}$

$f_x = e^{\sqrt{xy}} + x\underbrace{\left(e^{\sqrt{xy}}\right)\frac{1}{2}(xy)^{-\frac{1}{2}}}_{\text{Practise chain rule}} \cdot y$

$f_x = e^{\sqrt{xy}}\left(1 + \frac{\sqrt{xy}}{2}\right)$

$f_y = x \cdot e^{\sqrt{xy}} \cdot \frac{1}{2}(xy)^{-\frac{1}{2}} \cdot x$

3) $z = ye^{\frac{x}{y}}$ at $P(1,1)$

$= f(1,1) + f_x(x-a) + f_y(y-b)$

$f_x = ye^{\frac{x}{y}}\left(\frac{1}{y}\right) = e^{\frac{x}{y}}$

$f_y = 1 \cdot e^{\frac{x}{y}} + y\left(e^{\frac{x}{y}}\right) - \left(\frac{x}{y^2}\right)$

$f_y = e^{\frac{x}{y}}\left(1 - \frac{x}{y^2}\right)$

Tangent Eq: $e^{\frac{x}{y}}(x-1) + e^{\frac{x}{y}}\left(1 - \frac{x}{y^2}\right)(y-1)$

$= e^{\frac{x}{y}}\left[(x-1) + \left(1 - \frac{x}{y^2}\right)(y-1)\right]$ @ $(1,1)$

$= e\left[(0) + (0)\right]$

$= e \cdot x = ex$

4) $z = \sin(xy)$ at $x = 2$ and $y = 3\pi/4$

$f = z + f_x(x-a) + f_y(y-b)$

$f_x = \cos(xy)(y)$

$f_y = \cos(xy)(x)$

$= \sin(xy) + y\cos(xy)(x-2) + x\cos(xy)\left(y - \frac{3\pi}{4}\right)$

$z = f\left(2, \frac{3\pi}{4}\right) + f_x\left(2, \frac{3\pi}{4}\right)(x-2) + f_y\left(2, \frac{3\pi}{4}\right)\left(y - \frac{3\pi}{4}\right)$

$z = -1 + 0(x-2) + 0\left(y - \frac{3\pi}{4}\right)$

$z = -1$

5) $z = \ln(x^2+1) + y^2$ at $(0,3)$

$z_x = \frac{(2x)}{(x^2+1)}$

$z_y = 2y$

$= \left(\ln(x^2+1) + y^2\right) + \frac{2x}{x^2+1}(x-0) + 2y(y-3)$

$= 9 + 0 + 6(y-3)$

$= 9 + 6y - 18$

$= 6y - 9$

2) $f(x) = \frac{1}{a}e^{\frac{-x^2}{a^2}}$

$f_x = \frac{1}{a} \cdot e^{\frac{-x^2}{a^2}}\left(\frac{-2x}{a^2}\right) = -\frac{2x}{a^3} \cdot e^{\frac{-x^2}{a^2}}$

$f_a = -\frac{1}{a^2}e^{\frac{-x^2}{a^2}} + \frac{1}{a}e^{\frac{-x^2}{a^2}}\left(\frac{2x^2}{a^3}\right)$

6) $f(x,y) = x^2y$ and $V = (4,-3)$

$f_v = \nabla f \cdot \vec{v}$

$\nabla f = \langle f_x, f_y \rangle$

$\vec{v} = \frac{V}{\|V\|} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

$f_x = 2x \cdot y$

$f_y = x^2$

$f_v(2,6) = f_x(2,6) \cdot \frac{4}{5} + f_y(2,6) \cdot \left(\frac{-3}{5}\right)$

$= 24 \cdot \frac{4}{5} + 4\left(\frac{-3}{5}\right)$

$= \frac{84}{5}$

7) $f(x,y) = e^x\tan(y) + 2x^2y$

at the Point $(0, \pi/4)$

a) $(1,-1)$

$\vec{v} = \frac{(1,-1)}{\sqrt{2}}$

$\vec{v} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f_v = \left(f_x(x,y) + f_y(x,y)\right) \cdot \vec{v}$

$f_x = e^x \cdot \tan(y) + 4xy$

$f_y = e^x \cdot \sec^2(y) + 2x^2 \rightarrow \frac{e^x}{\cos^2(y)} + 2x^2$

$f_x\left(0, \frac{\pi}{4}\right) = 1$

$f_y\left(0, \frac{\pi}{4}\right) = 2$

The directional derivative is $f_v = \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$

b) $(1, \sqrt{3}) \longrightarrow \vec{v} = \frac{V}{|V|}$ $\quad \vec{v} = \frac{(1,\sqrt{3})}{\sqrt{1^2 + \sqrt{3}^2}}$

$\vec{v} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} + \frac{2\sqrt{3}}{2} = \frac{1+2\sqrt{3}}{2}$

8) $f(x,y) = x^2y^3$. At the point $(-1,2)$

a) vector in max ROC $\|\nabla f(a,b)\|$

$\nabla f(a,b) = \langle f_x(a,b) + f_y(a,b) \rangle$

$f_x = 2xy^3 \rightarrow -16$ $\quad \nabla F = -16i + 12j$

$f_y = 3y^2x^2 \rightarrow 12$

b) Min ROC: $-\nabla f = 16i - 12j$

c) $(-16 + 12j) \cdot (x, y) = 0$

$-16x + 12y = 0$

$-16(12) + 12(16) = 0$

$0 = 0$

$\therefore \vec{V} = (12, 16)$

9) $(-1, 1, 2)$ lies on $x^2 - y^2 + z^2 = 4$

Yes, it does lie on the given surface.

$(1, -1, 1) \rightarrow (x, y, z)$ s.t $x - y + z = 0$

$3 - (+1) - 2$

$3 - 1 - 2 = 0$

$0 = 0$

Eq Tp:

$f_x = 2x \quad f_x(-1,1,2) = -2$

$f_y = -2y \quad f_y(-1,1,2) = -2$

$f_z = 2z \quad f_z(-1,1,2) = 4$

Thus, gradient is $(-2, -2, 4)$. They are by default perp to level surfaces.

$-2(x+1) - 2(y-1) + 4(z-2) = 0$

10) $f(x,y,z) = x^2 + y^2 - xyz$

grad $F = \nabla f(xyz)$

$f_x = 2x - yz$

$f_y = 2y - xz$

$f_z = -xy$

grad $f = \langle (2x - yz)i + (2y - xz)j - (xy)K \rangle$

b) $f_x(2,3,1) = 11$

$f_y(2,3,1) = 4$

$f_z(2,3,1) = -6$

After plugging in $(2,3,1)$

grad $f(2,3,1) = i + 4j - 6k$

EQ: $(x-2) + 4(y-3) - 6(z-1) = 0$

$x + 4y - 6z = 8$

Homework 9 Questions

1) Find parametric Eq of the line passing through $(3,-2,2)$ and intersecting y axis @ $y=2$

Point $(0,2,0)$

1) Find direction vector

$A=(3,-2,2)$, $B=(0,2,0)$

$\vec{BA}=(3,-4,2)$

2) write in parametric form:

$$\begin{cases} x=3t \\ y=-4t+2 \\ z=2t \end{cases}$$

3) $r=3$ in yz Plane centered at Point $(0,0,2)$

$x=r\cos t$
$y=r\sin t$

has Equations: $x=0$
$y=3\cos t$
$z=3\sin t$

To move it of Center $(0,0,2)$ we get: $x=0$
$y=3\cos t$
$z=3\sin t+2$

7) Length of Curve.

7)
$$C=\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

$x=\cos(e^t)$, $y=\sin(e^t)$

$$\frac{dx}{dt}=-\sin(e^t)$$

$$\frac{dy}{dt}=\cos(e^t)$$

$$=\int_0^1\sqrt{(-\sin e^t)^2+(\cos e^t)^2}$$

$$=\int_0^1\sqrt{1}=(1)^{\frac{1}{2}}=e^t\Big|_0^1=e-1$$

b) The length of the arc of the unit Circle from the Points $\cos 1, \sin 1$ to $\cos e, \sin e$ are angles between angles $\theta=1$, $\theta=e$

Length is $e-1$.

2) $z=2x-3y+7$, passing through the $P(1,1,6)$

line perp to $\vec{n}=(2,-3,-1)\cdot(x,y,z)=0$

The normal vector to the Plane is already Perpendicular

$$\ell=\begin{cases} x=1+2t \\ y=1-3t \\ z=6-t \end{cases}$$

5) $x=5+7t$
$y=4+3t$
$z=-3-2t$

$\vec{d}=(7,3,-2)$
$\vec{P}=(5,4,-3)$

parallell to plane $2x-3y+5z=5$?

must share same normal vector

$\vec{n}=(2,-3,5)$

The following line is not Paralleli to the Plane due to different direction vectors.

6) $r_1(t)=((5-3t),(1+t),(2t))$

$r_2(t)=((2+6t),(2-2t),(2-4t))$

$\vec{d_1}=(-3,1,2)$
$\vec{d_2}=(6,-2,-4)$

Factor of -2
∴ same direction vector
∴ parallell

Point on $r_1(0)=(5,1,0)$

check if exists on other line

we know $2+6t=5$
$6t=3$
$t=\frac{1}{2}$ $\rightarrow$ $2-2(\frac{1}{2})=1$
$1=1$ ✓ $\rightarrow$ $2-4(\frac{1}{2})=0$
$2-2=0$
$0=0$

∴ Because the point exists on both lines, they are the same line.

4) $D=5$, Parallel to y axis, $d=2$, Parallel to z axis Centered at $(0,1,-2)$

General Elipse Equation in 2D:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Parametric form:

$x=a\cos t$
$y=b\sin t$

$$\frac{(a\cos t)^2}{a^2}+\frac{(b\sin t)^2}{b^2}=\cos^2 t+\sin^2 t=1$$

$D=5$, $a=5/2$
$d=2$, $b=1$
lies in yz Plane

$x=0$
$y=a\cos t$
$z=b\sin t$

then:

$$\begin{cases} x=0 \\ y=\frac{5}{2}\cos t+1 \\ z=\sin t-2 \end{cases}$$

8) Length of a Curve

$$r(t) = \langle 2t + \ln t + t^2 \rangle$$

$$\int_1^2 \sqrt{(2)^2 + \left(\frac{1}{t}\right)^2 + (2t)^2}$$

$$\int_1^2 \left(4 + \frac{1}{t^2} + 4t^2\right)^{\frac{1}{2}}$$

$$2\int_1^2 \left(t^{-2} + 4t^2\right)^{\frac{1}{2}} = 2 + \frac{\left(\frac{t^{-1}}{-1} + \frac{4t^3}{3}\right)^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^2 = 22.94 - 1.461$$

$$= 21.98$$

9) To find vectors Point toward the origin and have constant length,

a) $F(x, y) = -\frac{(xi + yj)}{\sqrt{x^2 + y^2}}$ $r = (x, y)$ Points outward

b) $F(x, y) = \frac{yi - xj}{\sqrt{x^2 + y^2}}$

A vector perpendicular to (x, y) can be $(-y, x)$.

$x'(t) = x(t)$

$\frac{dx}{dt} = x$ Isolate variables x and t

$\int \frac{dx}{x} = \int dt$ Integrate

$\ln|x| = t + C$ Solve for x

$x = e^t \cdot e^c$

$x'(t) = 3x(t)$

$\frac{dx}{dt} = 3dt$

$\int \frac{dx}{x} = \int 3\,dt$

$\ln|x| = 3t$

$x = e^{3t + c}$

10) (Flow of a vector field) $V = xi - yj$

a) Flow line passing through (1, 1) at $t = 1$

① $x'(t) = x(t)$ Set $x'(t)$ equal to

$y'(t) = -y(t)$ set $y'(t)$ equal to

② Solve $x'(t) = x(t)$

$\frac{dx}{dt} = dt \Rightarrow \ln|x| = t + C1 \Rightarrow x(t) = C_1 e^t$

$\frac{dy}{y} = -dt \Rightarrow \ln|y| = -t + c2 \Rightarrow y(t) = C_2 e^{-t}$

③ Apply $x(1) = 1, y(1) = 1$

$C1 = e^{-1}, C2 = e$

$x(t) = e^{-1} \cdot e^t = e^{t-1}$

$y(t) = ee^{-t} = e^{1-t}$

b) $x(t) \cdot y(t) = e^{t-1} \cdot e^{1-t} = 1$

$xy = 1$, Describes a hyperbola

As $t \to \infty$, $x \to \infty$, $y \to 0$

c) $x(t) \cdot y(t) = C$

$c > 0$: Hyperbolas in Q1 and Q3

$c < 0$: Hyperbolas in Q2 and Q4

Homework 10 Questions

1) $f(x,y) = (3, y+5)$ Line from $(0,0)$ to $(0,3)$

$$\int_c (3i + (y+5)j)\cdot dr$$

1) Parameterize:

$A = (0,0)$ $AB = (0,3)$

$B = (0,3)$

$r(t) = (x(t), y(t))$

$t\,[0,1]$

$x(t) = 0 + 0\vec{t} \longrightarrow x(t) = 0$

$y(t) = 0 + 3\vec{t}$ $y(t) = 3\vec{t}$

2) Plug into integral

$r(t) = (x(t), y(t))$

$r(t) = (0, 3t) \rightarrow r'(t) = 3$

$F\cdot dr = (3 + (3t+5))\cdot(0+3)\,dt$

$= 0 + 9t + 15$

$$\int_0^1 9t + 15\,dt = \frac{9t^2}{2} + 15t\Big|_0^1 = \frac{39}{2}$$

2) C - line from $(1,0,0)$ to $(5,0,0)$

$$\int_c (2x + 3y)\cdot dr$$

1) parameterize $\vec{AB} = (4,0,0)$

$x(t) = 1 + 4t$ $\frac{dx}{dt} = 4dt$ $\frac{dz}{dt} = 0$

$y(t) = 0$

$z(t) = 0$ $\frac{dy}{dt} = 0$ $t \in [0,1]$

$F\cdot dr = (2(1+4t) + 3(0))\cdot(4+0)$

$= (2 + 8t + 0)\cdot 4 + 0$

$= 8 + 32t$

$$\int_0^1 32t + 8\,dt = \frac{32t^2}{2} + 8t\Big|_0^1 = 24$$

3) C is line from $(3,1)$ to $(0,0)$

$$\int_c (2y^2 i + xj)\cdot dr$$

1) parameterize curve

$A = (3,1)$ $B = (0,0)$

$\vec{BA} = (3,1)$

Plug in B

$r(t) = (x(t), y(t))$

$x(t) = 0 + 3t$

$y(t) = 0 + 1t$

$t \in [0,1]$

$\frac{dx}{dt} = 3dt$

$\frac{dy}{dt} = 1dt$

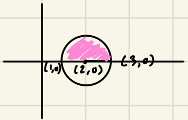
$F\cdot dr = ((2(t)^2)i + (3t))\cdot(3i + 1j)$

$$= \int_0^1 6t^2 + 3t = 2t^3 + \frac{3t^2}{2}\Big|_0^1$$

$= \frac{7}{2}$

Flip sign if you do BA

4)



radius = 1

Centered at (2,0)

Eq: $x^2 + y^2 + z^2 = r^2$

$x^2 + y^2 = 1^2$

$$\begin{cases} x(t) = 2 + \cos t \\ y(t) = \sin(t) \\ t \in [0, \pi] \end{cases}$$

$$\int_c (xi + yj)\cdot dr$$

$\frac{dx}{dt} = -\sin(t)$

$\frac{dy}{dt} = \cos(t)$

$F\cdot dr = ((2+\cos t)i + (\sin t)j)\cdot(-\sin(t) + \cos(t))$

$= -2\sin(t) - \sin(t)\cos(t) + \sin t\cos t$

$= -2\sin(t) - 1 + 1$

$$\int_0^\pi -2\sin(t) = -2\int_0^\pi \sin(t) = -2\cdot\cos(t)\Big|_0^\pi$$

$= -4$

5) C is the line segment from these 3 lines, 4 points

$(4,0,0)$ – A

$(4,3,0)$ – B

$(0,3,0)$ – C

$(0,3,5)$ – D

1) Parametrize each line segment on its own

1) Line $\overline{AB}$: $\vec{AB} = (0,3,0)$

$r_1(t) = (x(t), y(t), z(t))$

$x(t) = 4$

$y(t) = 3t$

$z(t) = 0$

$r_1(t) = (4i + 3tj + 0k)$ $0<t<1$

$r_1'(t) = 3$

2) Line $\overline{BC}$: $\vec{BC} = (-4, 0, 0)$

$x(t) = 4 - 4t$

$y(t) = 3$

$z(t) = 0$

$0<t<1$

$r_2(t) = (4 - 4ti + 3j + 0k)$

3) Line $\overline{CD}$: $\vec{CD} = (0,0,5)$

$x(t) = 0$

$y(t) = 3$ $0<t<1$

$z(t) = 5t$

$r_3 = (0i + 3j + 5tk)$

Continue on next page

5) continued -

$r_1(t) = (4i + 3tj + 0K) \quad 0<t<1$

$r_2(t) = (4-4ti + 3j + 0K) \quad 0<t<1$

$r_3 = (0i + 3j + 5tK) \quad 0<t<1$

$$\int_C F\cdot dr = \int_{C1} F\cdot dr + \int_{C2} F\cdot dr + \int_{C3} F\cdot dr$$

C1

$$\int_0^1 ((4)^2 + 3t)i + (3t)^3 j \cdot (0, 3, 0)$$

$$\int_0^1 3\cdot(3t)^3 = 3\int_0^1 27t^3 = \frac{81t^4}{4}\Big|_0^1 = \frac{81}{4}$$

C2

$$\int_0^1 \left(((4-4t)^2 + 3)i + (3)^3 j\right) \cdot (-4, 0, 0)$$

$$= \int_0^1 -76 + 128t - 64t^2 \quad = -76t + \frac{128t^2}{2} - \frac{64t^3}{3}\Big|_0^1$$

$$= \frac{-100}{3}$$

C3

$$\int_0^1 (0^2 + 3)i + (3)^3 j \cdot (0, 0, 5)$$

$$\int_0^1 0 = 0$$

Final answer : $\int_C F\cdot dr = \int_{C1} F\cdot dr + \int_{C2} F\cdot dr + \int_{C3} F\cdot dr$

$$= \frac{81}{4} - \frac{100}{3} + 0$$

$$= \frac{-157}{12}$$

6) $F = \text{grad}(x^2 + y^4)$

Using the FToLI, since F is the gradient of $f(x,y) = x^2 + y^4$

$$\int_C F\cdot dr = f(\text{start}) - f(\text{end})$$

$$\int_C F\cdot dr = f(0,2) - f(2,0) = 16 - 4 = 12$$

Circle in first quadrant, $x^2 + y^2 = 4$. Radius is 2, Points (2,0) and (0,2)

7) $F = \text{grad}(\sin(xy) + e^z)$

Using FToLI, only the starting and ending points matter.

$$\int_C F\cdot dr = f(\sqrt{2}, \sqrt{5}, 2) - f(0,0,0)$$

$$= \sin\sqrt{10} + e^2 - 1$$

8) $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \xrightarrow[\mathbb{R}^2]{} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Task: Find function f s.t. $\nabla f = F$

$\vec{F} = (\overset{F_1}{2xy}, \overset{F_2}{x^2})$ → Must be conservative $(\mathrm{Curl}(F) = 0)$

$\Rightarrow \frac{\partial F_2}{\partial x} \overset{??}{=} \frac{\partial F_1}{\partial y} \longrightarrow 2x \overset{??}{=} 2x$ HAPPY DAY ☺ → F conservative; f exists.

We know then that:

$$\frac{\partial f}{\partial x} = 2xy$$

$$\int$$

$$f(x,y) = \int 2xy \, dx$$

$$= 2\frac{x^2}{2}y$$

$$= x^2y + h(y) \rightarrow \text{why?}$$

$$\frac{\partial f}{\partial y} = \int x^2 \, dy$$

$$= x^2y$$

Potential function: $f(x,y) = x^2y + C$

9) F if grad $f = 2xyi + (x^2+8y^3)j$

① To find a Potential function, we must check if the function is conservative

$$f = (2xy, x^2 + 8y^3)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

$$\frac{\partial F_2}{\partial x} = x^2 + 8y^3$$

$$= 2x$$

$$\frac{\partial F_1}{\partial y} = 2x$$

$$f(x,y) = \int 2xy \, dx = \frac{\cancel{2}x^2}{\cancel{2}}y = x^2y + C(y)$$

$$f(x,y) = \int x^2 + 8y^3 \, dy = x^2y + \frac{8y^4}{4}$$

$$= x^2y + 2y^4 + K$$

---

10) $f(x,y,z) = x^2 + 2y^3 + 3z^4$

$\int_C F \cdot dr$, 4 line Segments

(4,0,0) to (4,3,0) to (0,3,0) to (0,3,5)

$F = \nabla f$

1) Because $F = \nabla f$

Using Fundamental Theorem of Line Integrals

$$\int_C F \cdot dr = f(\text{endpoints}) - f(\text{start points})$$

Start: (4,0,0) → Evaluate f(4,0,0)

End: (0,0,5) → Evaluate f(0,0,5)

$$f(4,0,0) = 4^2 + 2(0)^3 + 3(0)^4 = 16$$

$$f(0,0,5) = 0^2 + 2(0)^3 + 3(5)^4 = 1875$$

$$\int_C F \cdot dr = 1875 - 16$$

$$\int_C F \cdot dr = 1859$$

11) $F = 3x^2 + 4y^3$

$F = \nabla f$, where $f(x,y) = x^3 + y^4$

The Path C is the top half of the unit circle from (1,0) t
(-1,0).

1) Apply FTOLI

$$\int_C F \cdot dr = f(\text{end}) - f(\text{start})$$

$$f(-1,0) = -1$$

$$f(1,0) = 1$$

$$\int_C F \cdot dr = -1 - 1$$

$$\int_C F \cdot dr = -2$$