CISC 204

Well Formed Formula

Atom : $P, Q, r, S \ldots\ldots$ Typically Letters

Logical connectives:

- Negation ($\neg$) Reperesents "not"
- Conjunction ($\wedge$) Repercesents "and"
- Disjunction ($\vee$) Reperesents "or"
- Implication ($\rightarrow$) Reperesents "If... then"
- Biconditional ($\leftrightarrow$) Repelesents "If and only if)

By convention a formula is written according to the binding rules:

Hierarchy

$\neg$ tighter

$\vee, \wedge$ ⋮

$\rightarrow$ looser

$\leftrightarrow$

Rules for Forming a WFF:

1.) Any Single propositional variable (Atom) is WFF. Atomic

2.) If A is WFF, then $\neg A$ is also WFF. Negation

3.) If A and B are WFF then $(A \wedge B)$, $(A \vee B)$ $(A \rightarrow B)$ and $(A \leftrightarrow B)$

A WFF:

- is an atom
- if ϕ is a WFF then $(\neg \phi)$ is a WFF
- if ϕ is a WFF and ψ is a WFF then $(\phi \vee \psi)$ is a WFF
- if ϕ is a WFF and ψ is a WFF, then $(\phi \wedge \psi)$ is a WFF
- if ϕ is a WFF and ψ is a WFF then $(\phi \rightarrow \psi)$ is a W.F.F.

Example

we write $(\neg P)$

as $\neg P$

we write $(\neg P) \vee Q$

as $\neg P \vee Q$

Well Formed Formula: Syntactically correct expression in propositional logic.

Conjunction

$$\frac{\phi \wedge \psi}{\phi} \; ne1$$

↳ nan elimination type 1

$$\frac{\phi \wedge \psi}{\psi} \; ne2$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \; \wedge i$$

$p \wedge q + q \wedge p$

1.	$p \wedge q$	premise
2.	p	$1, \wedge e1$
3.	q	$1, \wedge e2$

$q \wedge p$

$(p \wedge q) \wedge r + p \wedge (q \wedge r)$

1.	$(p \wedge q) \wedge r$	premise
2.	$p \wedge q$	1, ne1
3.	q	2, ne2
4.	r	1, ne2
5.	p	1, ne1
6.	$q \wedge r$	3, 4 $\wedge$ 2
7.	$p \wedge (q \wedge r)$	5, 6, $\wedge$, 2

Steps of Proof can be different.

3 Rules for conjunction in natural deductions.

Conjunction overview: $(\wedge)$

- logical operator used to connects two propositions and is true if and only if both of the props are true
  (For $A \wedge B$ to be true, A must be true, B must be true.)

Conjunction Introduction Rule

$$\frac{P \quad Q}{P \wedge Q} \; (\wedge \text{ intro})$$

If P is its Raining = True
If Q is its cold = True
Then $P \wedge Q$ concludes its raining and cold

Conjunction Elimination Rules

- allows you to takeout one part of proposition

∧ - Elimination 1

Rule: If you have $P \wedge Q$ You can conclude P.

$$\frac{P \wedge Q}{P}$$

If you know its raining and cold you know its raining.

∧ - Elimination 2

Rule: If you have $P \wedge Q$ You can conclude P.

$$\frac{P \wedge Q}{Q}$$

2 Types of Reasoning

1). Forward: Start from premiss try to reach conclusion

2) Backward: Start from conclu to reach premisis

Proof:

1. $(P \wedge Q) \wedge R$ ∧ elim 1
2. R
3. $P \wedge Q$
4. Q
5. P
6. $Q \wedge R$
7. $P \wedge (Q \wedge R)$

Double negation

The rule of two negatives. Part of natural deduction

Example of starting with not not P and Q, is equal to P and Q.

Forward Example

$\neg\neg P \wedge Q \vdash P \wedge Q$

1. $\neg\neg P \wedge Q$	premise
2. $\neg\neg P$	$1, \wedge e^1$
3. Q	$1, \wedge e2$
4. P	$2, \neg\neg e^1$
5. $P \wedge Q$	$4, 3, \wedge i$

$\neg\neg P \wedge \neg\neg Q \vdash Q \wedge P$ → DEDUCE

1. $\neg\neg P \wedge \neg\neg Q$	premise
2. $\neg\neg P$	$1, \wedge e_1$
3. $\neg\neg Q$	$1, \wedge e2$
4. P	$2, \neg\neg e^1$
5. Q	$2, \neg\neg e^2$
6. $Q \wedge P$	$5, 4, \wedge$

Double negation elimination $\neg\neg P \to P$

Implication Elimination

$A \to B$

antecedent consequent

$$\frac{4 \qquad A \to B}{B} \to e$$

Modus Ponans

$$\frac{\neg B \qquad A \to B}{\neg B} MT$$

Modus Tollens

P → Q and P, conclude Q

P → Q and −Q, conclude −P.

Negation introduction

If assuming P leads to a contradiction, conclude ¬P.

$P, P \to (Q \wedge \neg\neg r) \vdash r$

1. P	Premise
2. $P \to (Q \wedge \neg\neg r)$	Premise
3. $Q \wedge \neg\neg r$	$1, 2, \to e$
4. $\neg\neg r$	$3, \wedge e2$
5. r	$4, \neg\neg e2$

$\neg P \to \neg Q, Q \vdash P$

1. $\neg P \to \neg Q$	premise
2. Q	premise
3. $\neg\neg Q$	$2, \neg\neg i$?
4. $\neg\neg P$	$3, 1, MT$
5. P	$4, \neg\neg e$

1. $P, P \to (Q \wedge \neg\neg R)$	
2. $Q \wedge \neg\neg R$	$1, \to e$
3. $\neg\neg R$	$2, \wedge e$
4. R	$3, \neg\neg e$

1. $Q \quad \neg P \to \neg Q$
2. $\neg\neg Q$
3. $\neg\neg Q \to \neg\neg P$
4. $\neg\neg P$
5. P

Negation

$\bot$ - Bottom

$$\frac{\phi \quad \neg\phi}{\bot}\neg e$$

$$\frac{\bot}{\phi}\bot e$$

$p \to r, p \wedge \neg q, \neg r \vdash q$

1.	$p \to r$	premise
2.	$p \wedge \neg q$	premise
3.	$\neg r$	premise
4.	p	2, $\wedge e_1$
5.	r	4,1 $\to e$
6.	$\bot$	5, 3, $\neg e$
7.	q	6, $\bot e$

Negation is a logical operation that reverses the truth

Ex. $\neg\neg P$ means P is true

any formula and its negation, negation elimination gives you bottom $\bot$

Negation Introduction:

If something leads to a contradiction, the opposite is true.

Modus Tollens

If P implies Q. Then not P, means not Q.

Implication Introduction

$$\frac{\boxed{\begin{matrix}A\\ \vdots\\ B\end{matrix}}}{A \to B}\to i$$

Assumption Box

Implication is transitive

$p \to q, q \to r \vdash p \to r$

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	p	assumption
4.	q	3,1 $\to e$
5.	r	4, 2 $\to e$
·	$p \to r$	3-5 $\to i$

Proving a Theorem

$\vdash p \to (q \to p)$

1.	p	assumption
2.	q	
3.	p	1, copy
4.	$q \to p$	2-3 $\to i$
5.	$p \to (q \to p)$	1-4 $\to i$

A theorem is a formula that is always True

Implication introduction is making an assumption box, antecedent is at the top conseauent at the bottom. Note, if something is in an outer assumption box, it is still in scope.

Week 2

Proof by contradiction

- Used to prove a statement is true by showing that assume the negation leads to contradiction

$\neg p \to p \vdash p$

1. $\neg p \to p$	premise
2. $\neg p$	assumption
3. p	2,1,→e
4. $\bot$	3,1 ¬e
5. p	2-4, PBC

Strategies for contradiction

$\neg(p \to q) \vdash p \land \neg q$

1. $\neg(p \to q)$	Premise
$\neg p$	
p	PBC
p	assumption
q	assumption
q	copy
$p \to q$	
$\bot$	¬e
$\neg q$	¬i
$p \land \neg q$	∧i

Disjunction Introduction

Disjunction Introduction allows you to introduce a disjunction (or Statement)

- If you know p is true, then you know p or any other statement q is true.

From p, conclude $p \lor q$

$p \to q \vdash p \to (r \lor q)$

1. $p \to q$	premise
2. p	assume
3. q	1,2,→e
4. $(r \lor q)$	3,∨i2
5. $p \to (r \lor q)$	2-4, →i

Law of Excluded Middle

Rules of inference.

$$\frac{}{\phi \vee \neg \phi}\ \text{L.E.M}$$

For any P, either P is true or it's negation ¬P is true

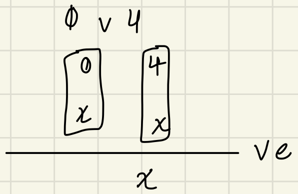
$P \vee \neg P$

$\vdash P \vee \neg P$

1. $\neg(P \vee \neg P)$	assumption
2. P	assumption
3. $P \vee \neg P$	2, ∨i 1
4. $\bot$	3, 1, ¬e
5. $\neg P$	2-4 ¬i
6. $P \vee \neg P$	5 ∨i 2
7. $\bot$	6, 1, ¬e
8. $P \vee \neg P$	PBC

Helpful with proofs by contradictions

Disjunction Elimination



Strategy that uses forward reasoning

Opens up 2 assumption boxes.

$\neg P \vee q \vdash P \to q$

1. $\neg P \vee q$	premise
2. P	assumption
3. $\neg P$	assumption
4. $\bot$	2, 3, ¬e
5. q	
6. q	assumption
7. q	3-5 6-6 ∨e
8. $P \to q$	2-7 →i

Ex 2

$P \to q \vdash \neg P \vee q$

1. $p \to q$	premise
2. $p \vee \neg p$	LEM
3. p	assumption
4. q	1, 3, →e
5. $\neg p \vee q$	4, ∨i 2
6. $\neg p$	assumption
7. $\neg p \vee q$	6, ∨i 1
8. $\neg P \vee q$	3-5, 6-7, ∨e

out of scope

Equivalence Logic

Law of excluded middle means disjunction elim right after.

Nesting Contradictions

$p \wedge \neg q \vdash \neg(\neg p \vee q)$

1.	$p \wedge \neg q$	premise
2.	p	$1, \wedge e_1$
3.	$\neg q$	$1, \wedge e_2$
4.	$\neg p \vee q$	assumption
5.	$\neg p$	assumption
6.	$\bot$	$2,5\ \neg e$
7.	q	assumption
8.	$\bot$	$3,7\ \neg e$
9.	$\bot$	$5\text{-}6, 7\text{-}8, \vee e$
10.	$\neg(\neg p \vee q)$	$4\text{-}9\ \neg i$

While attempting first negation you may assume another and its negation.

This leads to a second contradiction.

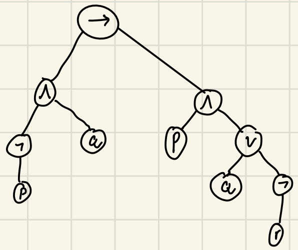
Implication Elimination P2

1.	T	premise
2.	$(Q \to T) \to (T \to R)$	premise
3.	Q	assumption
4.	T	1, copy
5.	$Q \to T$	$3\text{-}4 \to z$
6.	$T \to R$	$5, 2 \to e$
7.	R	

Using Implication partway through proof.

Parse Trees

$\neg P \wedge q \to p \wedge (q \vee \neg r)$



A visual reperesentation used in formal logic. To show structure of the formula.

1.) Nodes - reperesents a component of the formula such as an operator $(\wedge, \vee, \to)$

2.) Branches - show how components are combined

3). Root - top most node reperesents whole formula.

Propositional Semantics

Truth value : $\{T, F\}$

Model: assignment to a truth value to every atom

Truth Table

ϕ	$\neg\phi$
T	F
F	T

And Table

ϕ	ψ	$\phi\wedge\psi$
T	T	T
T	F	F
F	T	F
F	F	F

OR Table

ϕ	ψ	$\phi\wedge\psi$
T	T	T
T	F	T
F	T	T
F	F	F

Semantic Entailment :

Recall Sequent, ϕ, $\vdash$, ψ

Semantics that relate premises to conclusion.

Γ is a set of n W.F.F

$\{n_1, n_2, n_n\}$

About the meaning of the Statements. (Theres no way this is true and this conclusion is false)

A Semantic entailment holds is defined as $\Gamma \models \psi$. For every model in which every $\phi \in \Gamma$ evaluates to T.

1) $(p \wedge q) \models p$?

p	q	$p\wedge q$	p
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

Jape Notes:

- Contra (constructive) - used when you have $\bot$, can conclude anything
- contra (classical) - assumes the negation of smth. Then try to deduce bottom.
- ¬ elim to deduce bottom, if you have P and not P.
- To use hyp, must be like

  –B| ∨ q
  R ∨ q

- Law of excluded middle is ussually followed by disjunction elim
- Backwards conjunction intro - $P \wedge q$ and you have P and q
- If you have ¬P and p and bottom select both and then get rid of it using ¬ elim.

Week 3

Mathematical logic:

Formally prove Statements

A1: $(\phi \rightarrow (\psi \rightarrow \phi))$

A2: $((\phi \rightarrow (\psi \rightarrow x)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow x)))$

A3: $((\neg \psi \rightarrow \neg \phi) \rightarrow ((\neg \psi \rightarrow \phi) \rightarrow \psi))$

One Rule $\rightarrow e$ modus Ponens

Formalize methods of proof.

Consistency means no contradictions

Completeness

Let $\Gamma = \{ \phi_1, \phi_2 \dots \phi_n \}$

$(r \models 4) \rightarrow (F \vdash 4)$

Consider

$(P \rightarrow a) \vee (a \rightarrow P)$

P	q	$(p \rightarrow a) \vee (a \rightarrow P)$
T	T	T
T	F	T
F	T	T
F	F	T

$\vdash (p \rightarrow a) \vee (a \rightarrow P)$

1. $p \vee \neg P$ L.E.M

> P assumption
>
> $(P \rightarrow a) \vee (a \rightarrow P)$ $\vee i$

> $\neg P$ assumption
>
> $(p \rightarrow a) \vee (a \rightarrow p)$

$(p \rightarrow a) \vee (a \rightarrow p)$

Metalogic - Concept

Alphabet

$A = \{ 'p', 'a', 'r', \dots 'p_1', 'p_2' \dots \}$

operators

$\Omega = \{ '(', ')', '\neg', '\wedge', '\vee', '\rightarrow', '\bot', '\vdash' \}$

$T(\phi) \rightarrow F('(' \neg \phi ')')$

⋮

Represent

Let $\Gamma = \{ \phi, q_2 \cdots q_n \}$

$\Gamma \vdash 4$

Correct

$(\Gamma \vdash 4) \rightarrow (\Gamma \models q)$

sound

a logical theory that is Consistent and Correct.

Modus Ponens (Implication Elimination)

-Rule that allows you to conclude the Consequent of an implication if you know the antecedent is true. Implication Elimination

Ex. $P \to Q$

So if P is true, we can conclude Q

Modus Tollens

Deny the atecedent of an implication if the consequent is false, basically modus Ponens backwards.

Ex. If $P \to Q$, then we have $\neg Q$

Then we conclude $\neg P$.

Negation Introduction

-used for Proof by contradiction

$$\frac{\boxed{\begin{array}{c} A \\ \vdots \\ \bot \end{array}}}{\neg A}\ \neg i$$

Recall

$$\frac{A \to B \quad \neg B}{\neg A}\ \text{M.T}$$

$P \to Q, \neg Q \vdash \neg P$

1.	$P \to Q$	Premise
2.	$\neg Q$	premise
3.	P	assumption
4.	Q	$3,1 \to e$
5.	$\bot$	
6	$\neg P$	$3\text{-}5\ \neg i$

$\bot$ Bottom symbol can be used when you have P and not P.

Jape Notes:

- contra (constructive) - used when you have $\perp$, can conclude anything
- contra (classical) - assumes the negation of smth. Then try to deduce bottom.
- $\neg$ elim to deduce bottom, if you have P and not P.
- To use hyp, must be like $-B1 \lor Q$
  $R \lor Q$

- Law of excluded middle is ussually followed by disjunction elim
- Backwards conjunction intro - $P \land Q$ and you have P and Q
- If you have $\neg P$ and P and bottom select both and then get rid of it using $\neg$ elim.
- To unify B1 and S for example they must have the same form. (ctrl + alt)

Week 4

Semantics- Definitions

A model $\mathcal{M}$ is an assignment of T/F to every letter $P, Q, R \ldots$

Overload Symbol $\vDash$

$\mathcal{M} \vDash \psi$ is defined as
ψ evaluates to T under $\mathcal{M}$

$\Gamma = \{\phi_1, \phi_2 \ldots \phi_n\}$

For all $\mathcal{M}$ such that ever $\phi \in \Gamma$ evaluates to T, the formula ψ evaluates to T.

Tautology is a formula that is alway true

Satisfaction: Formula is Satisfiable

Semantic Equivalence:

$\phi \equiv \psi$ is defined as they semantically entail each other

$(p \wedge q) \equiv p \rightarrow (q \rightarrow r)$

using Truth table.

de Morgan Rules

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

Semantic Equivalence

Semantic Satisfiability

Consider : $(\neg\psi)$
Does $\vDash(\neg\psi)$ hold?
Tautology?

If $\vDash(\neg\psi)$ holds, then
$\forall_{\mathcal{M}} \ \mathcal{M} \vDash (\neg\psi)$

If the negation of a formula is a tautology, then the negation of that formula is true.

Disjunctive Tautologies

when does a tautology holds

Ex. $P \vee \neg P$

Extension: $P \vee Q \vee \neg r \vee \neg P$

Set of literals $\mathcal{L}$

Set of disjuncts of literals D

$(q \in E) \rightarrow (q \in D)$

Fact: a disjunct $D \in \mathcal{D}$ is a tautology if and only if
$\exists_i \ \exists_J, \exists_{L \in \mathcal{L}} \ (L_i = L) \wedge (L_J = \neg L)$
for $D = L_1 \vee L_2 \vee \ldots \vee L_m$

Truth Table : 2^n rows

Literal method : m literals

CNF Semantics (Conjunctive Normal Form)

Recall: A – atoms

$\mathcal{L}$ - literals are atoms or the negation of atoms

D - Disjuncts of literals

Define Conjuncts set C.

$$(D \in D) \rightarrow (D \in C)$$

$$((D \in D) \wedge (C \in C)) \rightarrow (D \wedge C) \in C)$$

Ex: $(P \vee r) \wedge (P \vee \neg r) \wedge (\neg P \vee r)$

Not: $(P \vee r) \wedge (\neg(P \vee q) \vee r)$

conjunctive normal form should describe truth table.

Given: Truth table for formula

For each row in which the formula evaluates to false write disjunction directly of the negation

p	q	φ	$\neg\varphi$
T	T	T	F
T	F	T	F
F	T	F	T
F	F	T	F

write $\neg\varphi$ as

$\neg P \wedge \neg q$

write φ as $\neg(\neg\varphi)$

$\neg(\neg P \wedge q)$

$\equiv \neg\neg P \vee \neg q$

$\equiv P \vee \neg q$

CNF - Syntax

φ is satisfiable iff $\neg\varphi$ is not true.

Recall: $\models \varphi$ iff $\vdash \varphi$

Negative Normal Form (NNF)

φ in which negation is <u>only</u> of a literal.

This 3 Step Syntactic alg is much faster than 2^n truth table.

3 pass algorithm

Pass 1: implication free

Equivalent: $A \rightarrow B \equiv \neg A \vee B$

Pass 2: NNF

Equivalent as de Morgan's Laws

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

Pass 3: Distribute

For $A \vee B$ use

Equivalent: say

$A \equiv A_1 \wedge A_2$

$A \vee B \equiv (A_1 \vee B) \wedge (A_2 \vee B)$

Auxillary Variables

Propositions to help a sub formula

$(P \lor Q)$

P	Q	$P \lor Q$	x_0
F	F	F	F
F	T	T	T
T	F	T	T
T	T	T	T

Iff and only if

$x_0 \leftrightarrow (P \lor Q)$

$x_0 \rightarrow (P \lor Q)$

$\neg x_0 \lor P \lor Q$

$(P \lor Q) \rightarrow x_0$

$\neg(P \lor Q) \lor x_0$

$(\neg P \land \neg Q) \lor x_0$

$(\neg P \lor x_0) \land (\neg Q \lor x_0)$

$(P \rightarrow x_0) \land (Q \rightarrow x_0)$

$x_1 \leftrightarrow (P \land Q)$

$x_1 \rightarrow (P \land Q)$

$\neg x x \lor (P \land Q)$

$(\neg x_1 \lor P) \land (\neg x_1 \lor Q)$

$(P \land Q) \rightarrow x_1$

$\neg (P \land Q) \lor x_1$

$\neg P \lor \neg Q \lor x_1$

$x_2 \leftrightarrow \neg P$

$x_2 \rightarrow \neg P$

$\neg P \rightarrow x_2$

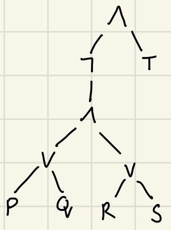
$\neg P \lor x_2$

Tseitin Encoding

Techiqne to go from boolean formula and convert to CNF.

$\neg((P \lor Q) \land (R \lor S)) \land T$

- m+n time complexity instead of m×n
- Breaks down complex formulas into subformulas then assigns variables which reperesent sub formula



$x_1 \leftrightarrow (P \lor Q)$

$x_2 \leftrightarrow (R \lor S)$

$x_3 \leftrightarrow x_1 \land x_2$

$x_4 \leftrightarrow \neg x_3$

$x_5 \leftrightarrow x_4 \land T$

$x_2 \rightarrow (R \lor S)$

$(\neg x_2 \lor R \lor S)$

$(R \lor S) \rightarrow x_2$

$\neg (R \lor S) \lor x_2$

$(\neg R \land \neg S) \lor x_2$

$(\neg R \lor x_2) \land (\neg S \land x_2)$

Metalogic

looking at logic at a higher level

Consistency - No way to derive at a contradiction (no $\bot$)

Correctness - Everything we can prove is semantically entailed

$$(\Gamma \vdash \varphi) \rightarrow (\Gamma \vDash \varphi)$$

Soundeness

Completeness - Everything semantically entailed is provable

Rules for NNF

1) Remove Implications $P \rightarrow Q$ with $\neg P \vee Q$

2) Push negations inward $\neg(p \wedge q) = (\neg p \vee \neg q)$
$\neg(p \vee q) = (\neg p \wedge \neg q)$

3) Simplify double negation $\neg\neg p = p$

4) negations can only be in front of variables

Rules for CNF

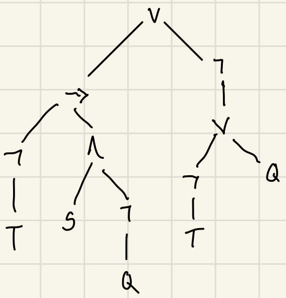
① get into nnf

② Distribute the OR

③ Final Formula should be just AND's seperating terms.

S5) $S5 = ((\neg T \to (S \land \neg Q)) \lor \neg(\neg T \lor Q))$

Parse Tree



Converting S5 to NNF

1. Starting Formula: $((\neg T \to (S \land \neg Q)) \lor \neg(\neg T \lor Q))$

2. Replace Implications ($A \to B \equiv \neg A \lor B$):

$$((\neg\neg T \lor (S \land \neg Q) \lor \neg(\neg T \lor Q))$$

3. Simplify double negation:

$$(T \lor (S \land \neg Q)) \lor \neg(\neg T \lor Q)$$

De Morgan's: $\neg(A \lor B) = (\neg A \land \neg B)$

4. De Morgan's law:

$$(T \lor (S \land \neg Q)) \lor (T \land \neg Q)$$

Rules for NNF

1) Remove Implications $P \to Q$ with $\neg P \lor Q$
2) Push negations inward $\neg(P \land Q) = (\neg P \lor \neg Q)$
$\neg(P \lor Q) = (\neg P \land \neg Q)$
3) Simplify double negation $\neg\neg P = P$
4) negations can only be in front of variables

Converting S5 to CNF

1. Starting formula (NNF):

$$(T \lor (S \land \neg Q)) \lor (T \land \neg Q)$$

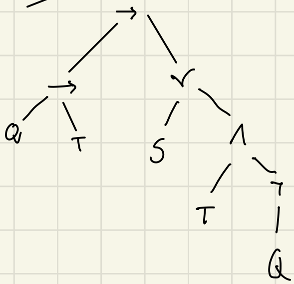
2. Distribute the OR $(A \lor (B \land C)) \equiv (A \lor B) \land (A \lor C)$

$$((T \lor S) \land (T \lor \neg Q)) \lor (T \land \neg Q)$$

$$(T \lor S) \land (T \lor \neg Q) \land (T \land \neg Q)$$

S6: $((Q \to T) \to (S \lor (T \land \neg Q)))$

Parse Trees



Converting S6 NNF

1.) Starting Formula: $((Q \to T) \to (S \lor (T \land \neg Q)))$

2. Replace Implications ($A \to B \equiv \neg A \lor B$):

$$(\neg Q \lor T) \to (S \lor (T \land \neg Q))$$

$$\neg(\neg Q \lor T) \lor (S \lor (T \land \neg Q))$$

3.) De Morgan's law:

$$(Q \land \neg T) \lor (S \lor (T \land \neg Q))$$

Converting S6 to CNF

1. Starting formula (NNF):

$$(Q \land \neg T) \lor (S \lor (T \land \neg Q))$$

2. Distribute the OR $(A \lor (B \land C)) \equiv (A \lor B) \land (A \lor C)$

$$(Q \lor S \lor T) \land (\neg T \lor S \lor T) \land (\neg T \lor S \lor \neg Q)$$

Week 5 - Motivation for Predicate Logic

Consider: Syllogism of Aristotle

Adding predicates

Map of an object to T or F.

$H(x)$ means x is human

$M(x)$ means x is mortal

$S(x)$ means x is socratis

Try

$H(x) \rightarrow M(x)$

$S(x) \rightarrow H(x)$

$S(x) \rightarrow M(x)$

Adding quantifiers

"For all" $\forall$

"There exists" $\exists$

$\forall x(H(x) \rightarrow M(x))$

$\exists x(S(x) \wedge H(x))$

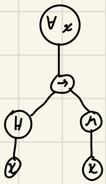
$\therefore$

Predicate logic introduces quantifiers and predicates that allow us to reason abt individuals.

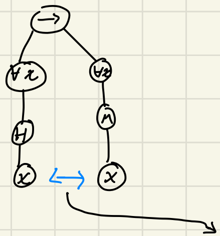
Parsing Predicates

Parsing means breaking it down and analyzing

$\forall x(H(x) \rightarrow M(x))$



Try: $\forall x\, H(x) \rightarrow \forall x . M(x)$ → If Everything is Human. Then Everything is mortal



With predicate logic, you cant be sure these 2 x's are equal

Try:

$\forall x(H(x) \wedge M(x))$

Everything is both human and mortal

Try:

$\exists x(S(x) \rightarrow H(x))$

Formal Language

Variables: x, y, z, w

Constant: a, b, c

nullery, no arguments

Function

zero or more arguments, maps to objects

$f(x)$ $f(x,y)$ $f(t_1, t_2, t_3)$

arguments and terms

Predicate: maps zero or two terms to a set. If it maps to a set can be T or F.

Extentional Definition

$\forall$ - Universal quantifier

$\exists$ - Existential quantifier.

Week 6

Universal Elimination

Rule: if you have a universally quantified formula (For all x in formula) you can derive $\phi(t)$ where t is any term free for x in ϕ

Notation: $$\frac{\forall x\, \phi}{\phi[t/x]}$$

Forward Reasoning

Ex

$P(a), \forall x(P(x) \to Q(x)) \vdash Q(a)$

1.	$P(a)$	Prem
2.	$\forall x(P(x) \to Q(x))$	Prem
3.	$P(a) \to Q(a)$	$\forall x$ e, 2
4	$Q(a)$	$\to$ e, 1, 3

Universal Introduction

Rule: if you prove a formula $\phi(x)$ holds for any fresh variable z. Then you can generalize this for $\forall x\, \phi(x)$.

Notation: $$\frac{\boxed{\begin{array}{l} z \\ \quad\vdots \\ \quad\phi[z/x] \end{array}}}{\forall x\, \phi}\ \forall x\, z$$

Backward Reasoning

Ex $\forall x(P(x) \to Q(x)), \forall x\, P(x) \vdash \forall x\, Q(x)$

1.	$\forall x(P(x) \to Q(x))$	Prem
2.	$\forall x\, P(x)$	Prem
3.	z	
4.	$P(z) \to Q(z)$	$\forall x$e 1
5.	$P(z)$	$\forall x$e 2
6.	$Q(z)$	$\to$e 5,4
7.	$\forall x\, Q(x)$	$\forall x$i

Existential Introduction

Rule: If a formula holds for some specific t. You can infer that another variable x such that it holds

Notation: $$\frac{\phi[t/x]}{\exists x\, \phi}$$

The term t must be free for x in ϕ.

Backward Step

Ex

$\forall x\, P(x) \vdash \exists y\, P(y)$

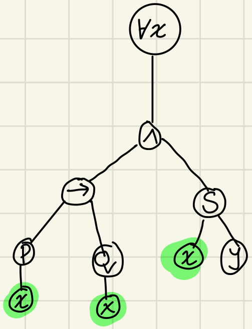
1.	$\forall x\, P(x)$	prem	
2.	$P(t)$	$\forall x$e 1	→ introduces new var
3.	$\exists y \cdot P(y)$	$\exists x$i 2	→ encapsulates it

Introduction to Scope

The scope of a quantifier is "what it applies to".

Example

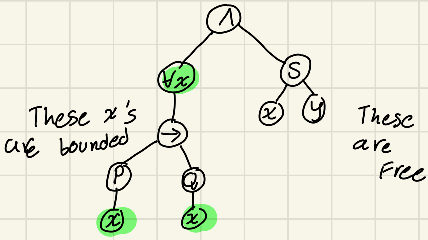
$$\forall x((P(x)) \rightarrow Q(x)) \wedge S(x,y))$$



In this case all the x are bounded by $\forall x$

The y is free variable

$$(\forall x(P(x) \rightarrow Q(x))) \wedge (S(x,y))$$



Scope and Binding

Concept: strict subformula

that is $\forall x(P(x) \wedge Q(x))$ means $P(x)$ is not the whole expression

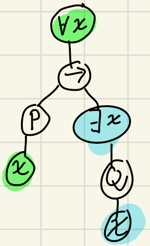
For $(Q\, x\, \phi)$ the scope of Q is:

ϕ minus each sub formula.

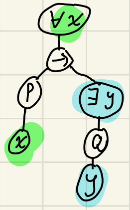
Rule

In the parse tree the quantifier has control over the variable unless you see another quantifier.

Example $\forall x(P(x) \rightarrow \exists x\, Q(x))$



Equivalent: $\forall x(P(x) \rightarrow \exists y\, Q(y))$



This is better form than to redefine your x variable.

Binding

A variable is bound if it is linked to a quantifier within it's scope.

A variable is free if it is not within the scope of a quantifier.

Scope of a quantifier is where it applies in the formula

Free for x in formula: if we substitute t for x the erm t will not accidentally be bounded by any quantifier in formula.

Fresh variable: variable that hasn't been used

Variable Capture: when you sub in but it changes the bounds on stuff. So you need a fresh variable.

Existential Elimination

Rule: Allows you to infer a statement, given some ∃, you can assume an argument. The instance must be treated like a fresh variable.

Notation:

$$\dfrac{\exists x\,\phi \qquad \boxed{\begin{array}{ll} t & \phi[t/x] \\ & \vdots \\ & x \end{array}}}{x}\ \exists x e$$

Basically add a fresh var on top then break it down.

Ex

$$\exists x\,(P(x) \land Q(x)) \vdash \exists x\,P(x)$$

1.		$\exists x(P(x) \land Q(x))$	prem
2.	z	$P(z) \land Q(z)$	assum
3.		$P(z)$	$\land e_1$, 2
4.		$\exists x\, P(x)$	$\exists x$ 2 3
5.		$\exists x\, P(x)$	$\exists x e$ 1, 2-4

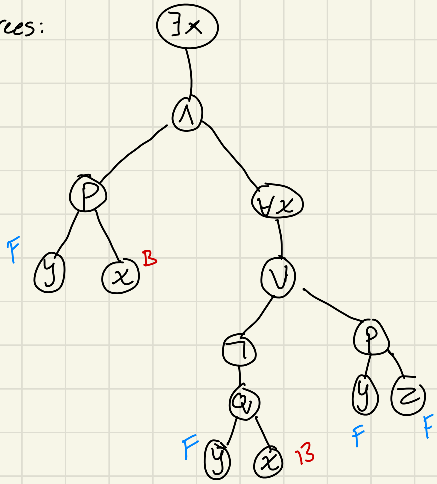
Forward Step:

- Existential Elim makes a new variable for you.

Example

$$\exists x\,(P(y,x) \land \forall x(\neg Q(y,x) \lor P(y,z)))$$

Parse Trees:



Jape Specific Notes

- to perform universal elimination, you click the statement and the actual i variable, forward step
- for implication elim, first click the first part of the implication
    - universal introduction is a backwards step that assumes an actual i and then the other part of the formula.
    - Existential introduction is a backwards step, select Ex part and the actual i
    - Existential Elim, introduces an actual i, new var.
    - Disjunction Elim, creates two cases, left and right of the V logo.
    - When you have ⊥, click it and whatevers under it and do contra constructive
    - If you have s and S(x), do hyp.
    - If you do negation of smth and click ⊥ then it can be brought into the assumption box

Week 7

Combining universal and Existential

All human's are mortal
Something is human
∴ Something is mortal

Ex $\forall x (H(x) \to M(x)), \exists x H(x) \vdash \exists x \cdot M(x)$

1. $\forall x (H(x) \to M(x))$ prem
2. $\exists x H(x)$ prem

> 3. w $H(w)$ assume
> 4. $H(w) \to M(w)$ $\forall$ei
> 5. $M(w)$ $\to$e 3,4
> 6. $\exists x (M(x))$ $\exists$i 5

7. $\exists x (M(x))$ $\exists x$ei 3-6

Universal quantifies and conjunctions

If you have universal quantifier applied to conjunction

$$\begin{tikzcd}
 & \forall x (P(x) \land Q(x)) \arrow[dl] \arrow[dr] & \\
\forall x (P(x)) & & \forall x (Q(x))
\end{tikzcd}$$

Ex $\forall x \cdot P(x) \land \forall x Q(x) \vdash \forall x (P(x) \land Q(x))$

1. $\forall x P(x) \land \forall x Q(x)$ premise
2. $\forall x P(x)$ $\land$e1, 1
3. $\forall x Q(x)$ $\land$e2, 1

> 4. z assume
> 5. $P(z)$ $\forall x$e 2
> 6. $Q(z)$ $\forall x$e 3
> 7. $P(z) \land Q(z)$ $\land$i 5,6

8. $\forall x (P(x) \land q(x))$ $\forall x$i 4-7

Universal Quantifiers

- when you have multiple universal quantifiers, order does not matter.

$$\forall x \forall y P(x,y) \equiv \forall y \cdot \forall x (P(x,y))$$

Used for reordering

Ex $\forall x \forall y R(x,y) \vdash \forall y \forall x R(x,y)$

1. $\forall x \forall y R(x,y)$ Prem

> 2. z
> > 3. w
> > 4. $\forall y R(w,y)$ $\forall x$ei
> > 5. $R(w,z)$ $\forall x$e4
>
> 6. $\forall x R(x,z)$

7. $\forall y \forall x R(x,y)$ $\forall x$e 3-6.

Combining propositional and Predicate Deduction

- Mixing the two logics

EX

$\forall x(\neg P(x) \vee Q(x)) \vdash \forall x(P(x) \to Q(x))$

1.	$\forall x(\neg P(x) \vee Q(x))$	prem
2.	w	
3.	$\neg P(w) \vee Q(w)$	$\forall x$ e 1
4.	$\neg P(w)$	assume
5.	$P(w)$	assume
6.	$\bot$	$\neg$e 5,4
7.	$Q(w)$	$\bot$e 6
8.	$P(w) \to Q(w)$	
9.	$Q(w)$	assume
10.	$P(w)$	assum
11.	$Q(w)$	copy
12.	$P(w) \to Q(w)$	i 10-11
13.	$P(w) \to Q(w)$	$\vee$e 3, 4-8, 9-12
14.	$\forall x(P(x) \to Q(x))$	$\forall x$ i

Multiple Quantifiers

- order Matters
- nested Quantifiers

EX1

$\forall x \forall y(P(y) \to Q(x)) \vdash \exists y P(y) \to \forall x \cdot Q(x)$

1.	$\forall x \cdot \forall y \cdot (P(y) \to Q(x))$	prem
2.	$\exists y \; P(y)$	assumption
3.	$w \;\; P(w)$	$\forall x$e1
	z	$\forall x$e5
	$\forall \; P(y) \to Q(z)$	
	$P(w) \to Q(z)$	$\to$ e3,6
	$Q(z)$	$\forall$ i 4-7
	$\forall x \, Q(x)$	$\exists x$e 1, 3-8
	$\forall x \, Q(x)$	
	$\exists \; P(y) \to \forall x \, Q(x)$	$\to$i 2-9

Week 8

Semantics of Predicate Logic

- Meto logic set theory

Recall:

P	Q	ϕ
T	T	T
F	F	F
T	F	T
F	T	T

Universe of Discourse – all objects we are going to allow in a model.

Notation

Syntax	Semantics
function f	$f^{\mathcal{M}}$
Predicate P	$P^{\mathcal{M}}$
discourse	$\mathcal{A}$
model	$\mathcal{M}$
Set of functions	$\mathcal{F}$
Set of Predicates	$\mathcal{P}$

Basics of Semantics of Predicates

$\mathcal{A}$: non empty set

Map $P: t \to [T, F]$

General, Complicated

Extensional

- Set

$$P^{\mathcal{M}} = \{0, 2\}$$

$$\mathcal{A} = \{0, 1, 2, 3\}$$

$$Q^{\mathcal{M}} = \{1, 3\}$$

$$\forall x \left(P(x) \vee Q(x) \right)$$

$$R^{m} = \{(0,1), (1,2), (2,3)\}$$

Week 9

Semantics of Equality

At anytime in a Proof
You can say $t = t$

t_1 can be subbed for t_2
as long as its free for x.

Equality is "fixed"
- Can't define equality

Semantics of Functions

A function is a map into A

Nullery: f $f(\,)$ (a function of no input mapping to a single A)
Unary: $f: A \to A$ $f(t)$ (mapping each element t to another element $f(t)$ in A)
Binary: $f: A \times A \to A$ $f(t_1, t_2)$ (mapping pairs to a single value)
n-ary: $f: A^n \to A$ $f(t_1, t_2 \dots t_n)$ (n inputs to 1 output)

(Succesor Function) $S(t) = (t+1) \bmod 4$ which cycles through values in A.

$S(2) = 1$
$S(S(2)) = 2$ → recursive application

Semantics of Unbound Variables

The set Var is defined as the set of all unbound variables in a formula

Lookup Function, denoted $l: \mathrm{Var} \to A$ is introduced. This function maps each unbound variable in Var to a value in a domain A.
Bassically, lookup function assigns a value to an unbound variables.

Semantics Models:

A model is a structure that uses the symbols in a formal language.

$\mathcal{F}$: a set of function symbols
$\mathcal{P}$: a set of predicate symbols

A model m of $(\mathcal{F}, \mathcal{P})$ assigns meaning to these symbols

Semantic Models

4 Key components

1. A non empty set A: (The domain of discourse) (set of all possible outcomes)
2. If $f \in F$ is nullary then M includes a specific function f^M that maps to single A.
3. If $f \in F$ is n-ary, then M includes a specific function $f^M: A^n \to A$, mapping n objects in A to an object in A.
4. If $p \in P$ is an n-ary predicate, M includes a specific function $p^M: A^n \to \{T, F\}$

The point of a model is to make abstract statements, meaningful and determines True and Fals's within a domain.

Two Models of (F, P)

$F = \{i, s\}$
$P = \{Q, R\}$

Model M_1: Binary
$A = \{0, 1\}$
Binary function $f(x,y) \in F$
In this model $f^M(x,y)$ could logical AND. So theres a unary Predicate. If $x=1$, True, $x=0$ False

Model M_2: base-4
$A = \{0, 1, 2, 3\}$
The same binary function can reperesent modular addition mod 4.

$$f^M(2,3) = (2+3) \bmod 4 = 1$$

Translating English to Predicate Logic

Reminders

$\forall$ "for all"
$\exists$ for "at least one"
$\wedge$ and
$\vee$ or
$\rightarrow$ implies
$\neg$ Not

$\neg\forall$ is Equivalent to $\exists x$